# On economical construction of the transitive closure of a directed graph*

V. L. Arlazarov†, E. A. Dinič‡, M. A. Kronrod§, I. A. Faradžev¶

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1. For a directed graph $(H, \gamma)$, where $H$ is the set of vertices, and $\gamma$ is a mapping from $H$ onto itself ($^1$), algorithms are known for constructing the transitive closure $\Gamma$, with the number of operations for a generic graph being $O(n^3)$ ($^{2-4}$), where $n = |H|$. In ($^4$), the problem of constructing the transitive closure of a generic graph reduces to a sequence of three problems: the construction of the Herz graph[1] ($^5$) for the given graph, the transitive closure of the (acyclic) Herz graph, and the construction of the transitive closure of the given graph based on the transitive closure of its Herz graph; meanwhile, it has been shown that the first and third of these can be solved in $O(n^2)$ operations.

In this article an algorithm is constructed for the transitive closure of an **acyclic** graph in $O(n^3/\ln n)$ operations.

2. For the acyclic graph $(H, \gamma)$, let us consider partitioning $H$ into ranks $K_i$:

$$K_0 = \{h \in H : \gamma^{-1}h = \varnothing\}, \quad K_i = \{h \in H \setminus S_{i-1} : \gamma^{-1}h \subset S_{i-1}\},$$

where $S_i = \bigcup\limits_{j \le i} K_j$.

In ($^4$) an algorithm is presented for such a partition in $O(n^2)$ operations.

Let us denote by $\gamma_i$ and $\Gamma_i$ the mappings from $S_{i-1}$ onto $K_i$ generated by $\gamma$ and $\Gamma$, respectively, and by $G_i = \bigcup \Gamma_j$ the mapping from $S_{i-1}$ onto $S_i$. Then, clearly:

$$G_0 = \varnothing, \quad \Gamma_i = \gamma_i G_{i-1} \bigcup \gamma_i. \tag{1}$$

In this way the construction of the transitive closure is reduced to the triangular process of obtaining products of mappings.

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*`https://www.mathnet.ru/links/5e5f8b0e3cca0b69911b55aa906afe5f/dan35675.pdf`

†Institute for Information Transmission Problems of the Academy of Sciences of the USSR, Moscow

‡Institute of Control Sciences, Moscow

§M. V. Lomonosov Moscow State University, Moscow

¶Central Research Institute of Patent Information, Moscow

[1]Graph of Herz, the condensation graph obtained by collapsing all strongly-connected components

3. **Lemma** (M. Kronrod). *Let $A$, $B$, $C$ be sets, $|A| = p$, $|B| = q$, $|C| = r$, and $\alpha$, $\beta$ be multi-valued mappings $\alpha : A \to B$, $\beta : B \to C$.*

*Then the mapping $\beta\alpha : A \to C$ can be constructed in $O((p+q)qr/\ln q)$ operations.*

**Proof**. Let us partition $B$ into $[q/\ln q] + 1$ nonintersecting subsets $B_i$, so that $|B_i| \leq \ln q$. Let us denote $\alpha_i : A \to B_i$, $\beta_i : B_i \to C$, generated by $\alpha$ and $\beta$ respectively. Then, clearly:

$$\beta\alpha = \bigcup_i \beta_i \alpha_i. \tag{2}$$

Let us consider the set $M_i = \{m_{is}\}$ of all subsets of $B_i$ and the set $L_i = \{l_{is}\}$ isomorphic to it, where $l_{is} = \bigcup\limits_{b \in m_{is}} \beta_i b$. Having enumerated the elements of $B_i$: $b_{i0}$, $b_{i1}$, etc., let us order the elements of $M_i$ and $L_i$ in the following way: $s = \sum\limits_k 2^{j_k} \leftrightarrow m_{is} = \bigcup\limits_k b_{ij_k}$. It is clear that $l_{i0} = \varnothing$, $l_{i_2 j} = \beta_i b_{ij}$, and any other $l_{is}$ can be obtained by the union of two others with smaller indices. For example: $l_{is} = l_{ik} \bigcup l_{i,s-k}$; $s \neq 2^j$, where$k = \max\{2^j : 2^k < s\}$. Since $|L_i| \leq q$, and $|l_{is}| \leq r$, then $L_i$ can be constructed in $O(qr)$, and all the $L_i$ — in $O(q^2 r/\ln q)$ operations.

Furthermore, from the completeness of $M_i$ it follows that for $a \in A$, $\alpha_i a \in M_i$ and, consequently, $\beta_i \alpha_i a \in L_i$. In this way, after all the $L_i$ have been constructed, obtaining $\beta\alpha$ according to (2) takes $O(pqr/\ln q)$ operations.

4. **Theorem**. *For the acyclic directed graph $(H, \gamma)$ with $|H| = n$ the transitive closure $\Gamma$ can be constructed in $O(n^3/\ln n)$ operations.*

**Proof**. Let us apply the algorithm of the lemma to the construction of $\gamma_i G_{i-1}$ in (1). Since $\gamma_i$ acts from $S_{i-1}$ with $|S_{i-1}| < n$ in $K_i$ with $|K_i| = n_i$, and $G_{i-1}$ acts from $S_{i-2}$ with $|S_{i-2}| < n$ in $S_{i-1}$, the construction of $\gamma_i G_{i-1}$ can be done in $O(n^2 n_i/\ln n)$ operations. Summing over the ranks, we obtain the desired bound.

## Cited Literature


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