# Outline of the talk

1. Introduction
2. Stochastic embedding into trees
3. Bartal 96 and Padded decompositions
4. Online Metric Embeddings
5. Spanning trees and MPX
6. Minor Free Graphs

# Metric Embeddings into Trees and its Various Spin-offs

Arnold Filtser

Bar-Ilan University

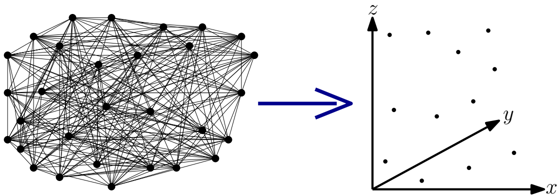
Dagstuhl Seminar 25212: Metric Sketching and Dynamic Algorithms for Geometric and Topological Graphs

May 19, 2025

# Metric Embeddings

## Embedding

$(X, d_X), (Y, d_Y)$ metric spaces. $f : (X, d_X) \to (Y, d_Y)$ is called an **embedding**.

$f$ : 

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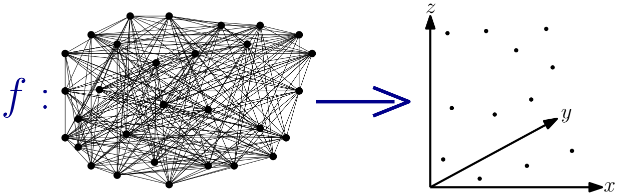


## Preserve (approxierly) properties of the original space:

- **Distances**
- Cuts, Flows
- Commute time
- Effective resistance
- Clustering statistics.
- etc.

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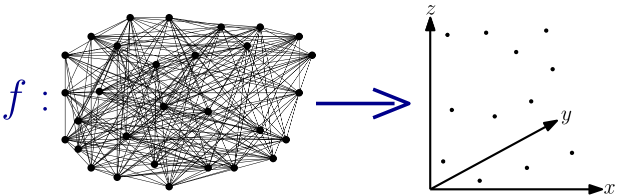


$f$ has **distortion** $t$ if:

$$\forall x, y \in X, \qquad d_X(x, y) \le d_Y(f(x), f(y)) \le t \cdot d_X(x, y) \ .$$

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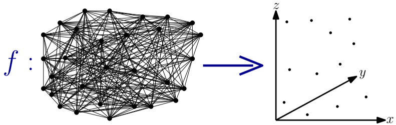


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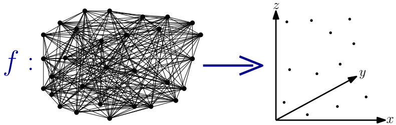


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So that we could run efficient algorithms on it...

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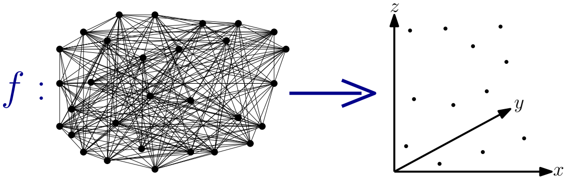


## Theorem ([Bourgain 85])

*Every n-point metric $(X, d_X)$ is* ***embeddable*** *into* <u>*Euclidean*</u> *space $(\mathbb{R}^d, \|\cdot\|_2)$ with* ***distortion*** *$O(\log n)$.*

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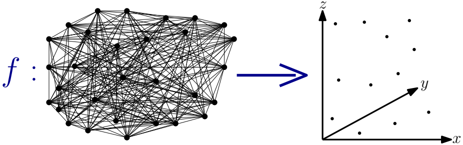


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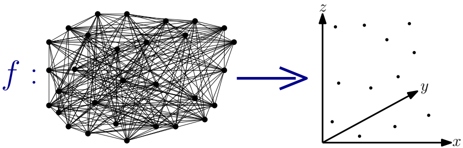
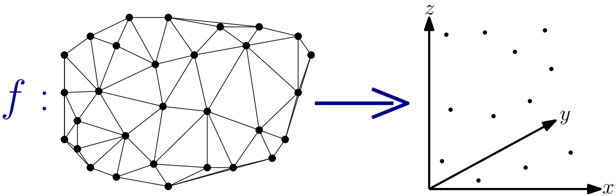



Applications:

- Approximation algorithms (e.g. **sparsest cut**, min graph bandwidth)
- Parallel computation (e.g. SSSP in MPC)
- Computational Biology (e.g. clustering and detecting protein seq.)
- etc.

## Theorem ([Rao 99])

*Every $n$-point **planar** metric $(X, d_X)$ (or fixed minor free) is **embeddable** into Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$ with **distortion** $O(\sqrt{\log n})$.*




Planar metric- the shortest path metric of a planar graph.

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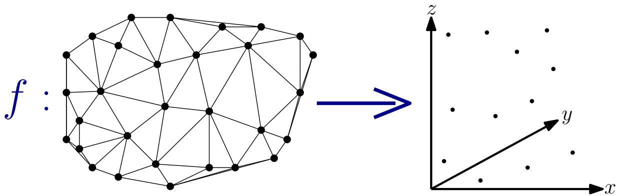


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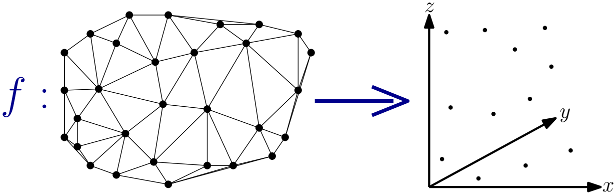


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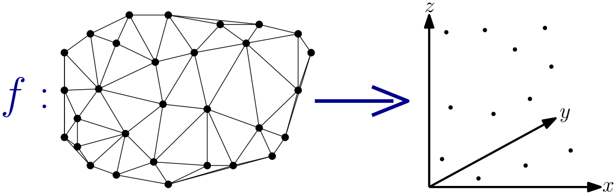


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## GNRS conjecture [Gupta, Newman, Rabinovich, Sinclair 04]

Every fixed minor free graph can be embedded into $\ell_1$ with constant distortion.

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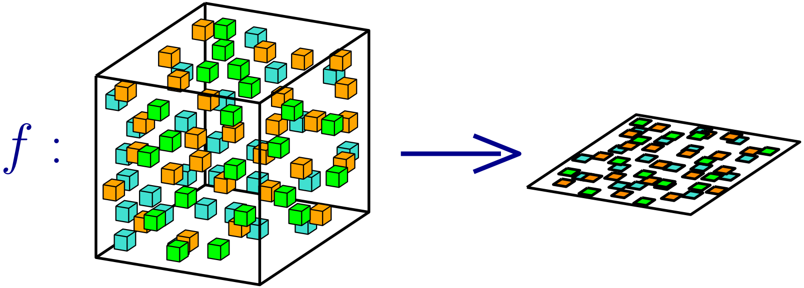

## Theorem ([Johnson, Lindenstrauss 84], Dimension Reduction)

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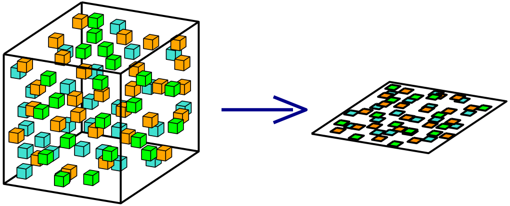
$f$ : 

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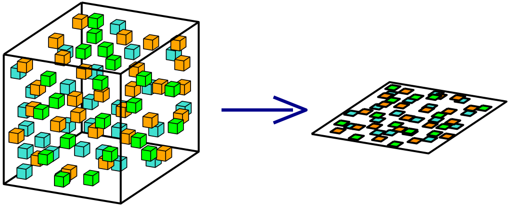

Applications:

- Speeding up-computation
- Clustering
- Nearest Neighbor Search
- Machine Learning
- etc.

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Tree is very **simple** and **desirable** target space.

Many NP-hard problems are easy on trees (using dynamic programming).

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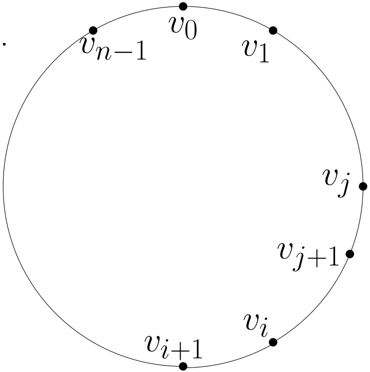


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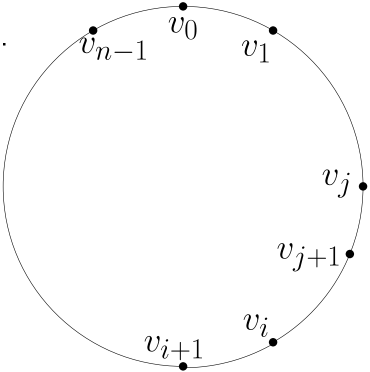


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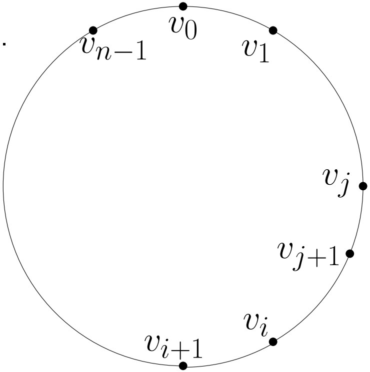


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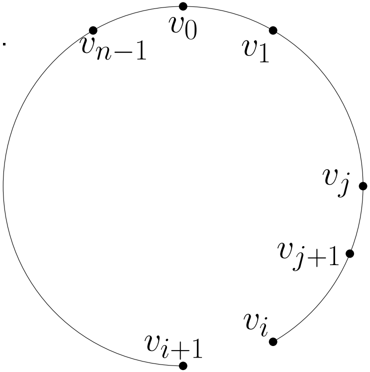


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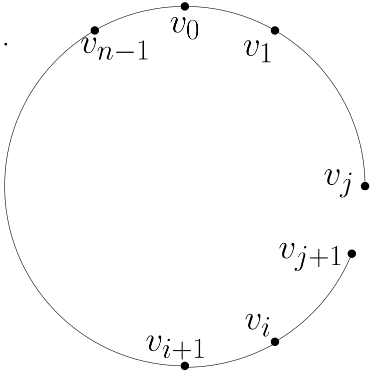


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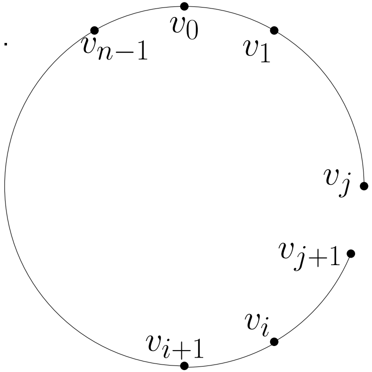


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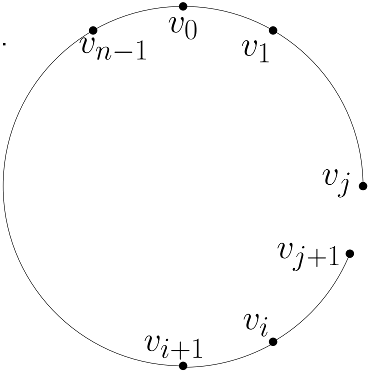


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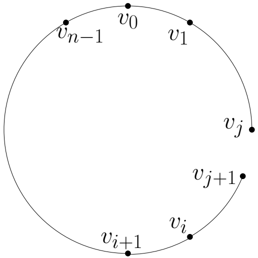


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By triangle inequality and linearity of expectation

$$\forall v_i, v_j, \quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(v_i,v_j)] = \sum_{q=i}^{j-1}\mathbb{E}_{T\sim\mathcal{D}}[d_T(v_q,v_{q+1 \pmod n})] \le 2\cdot d_{C_n}(v_i,v_j) \ .$$

# Stochastic Embedding into Trees

**Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])**

*Every $n$-point metric space $(X, d)$ embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $O(\log n)$.*

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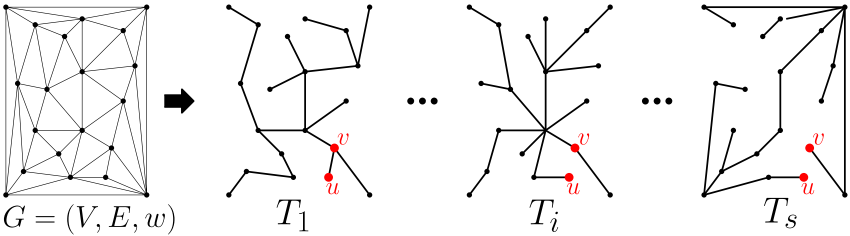

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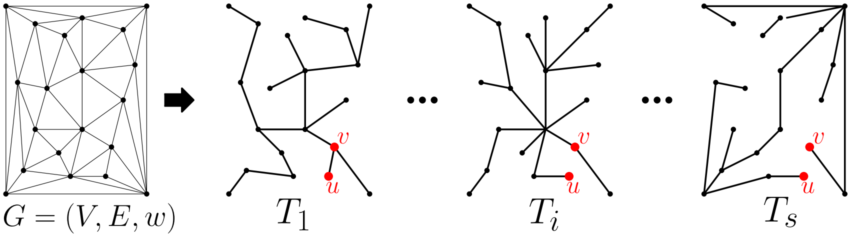


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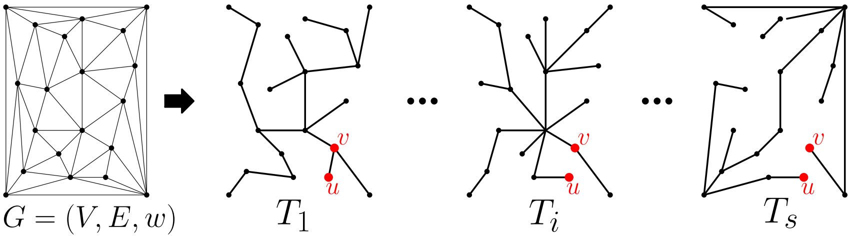


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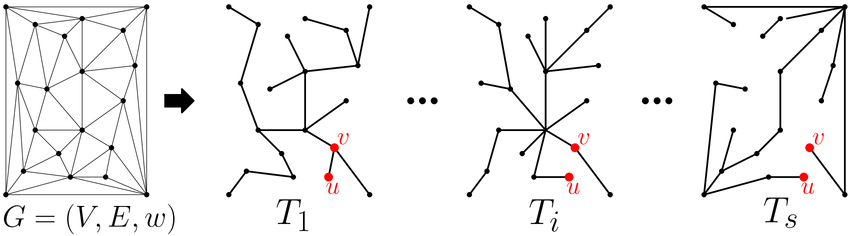


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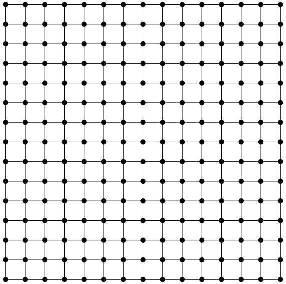
In fact, tight already for the $n \times n$ grid graph!



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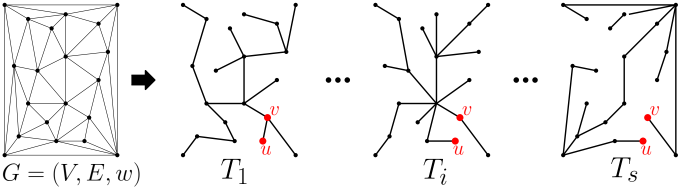


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Applications:

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- Online Algorithms.
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We will begin our tour of metric embeddings into trees with the classics: [Bartal 96]

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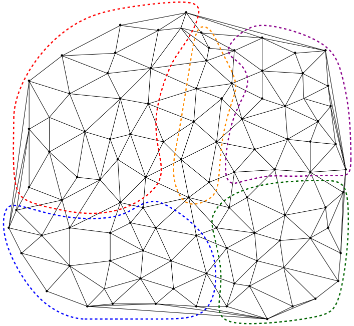
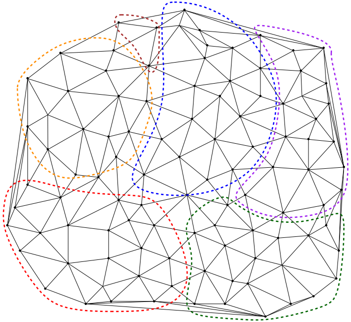
This one is based on random partitions of metric spaces.

## Definition (Padded Decomposition)

Given a metric space $(X, d_X)$ (or a weight graph $G = (V, E, w)$).
**Distribution** $\mathcal{D}$ over partitions of $G$ is $(\beta, \Delta)$-padded decomposition if:





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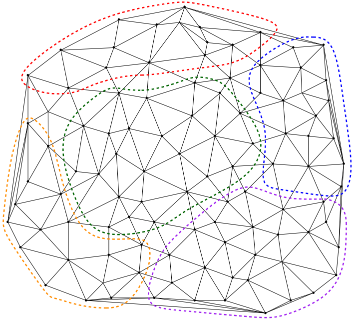

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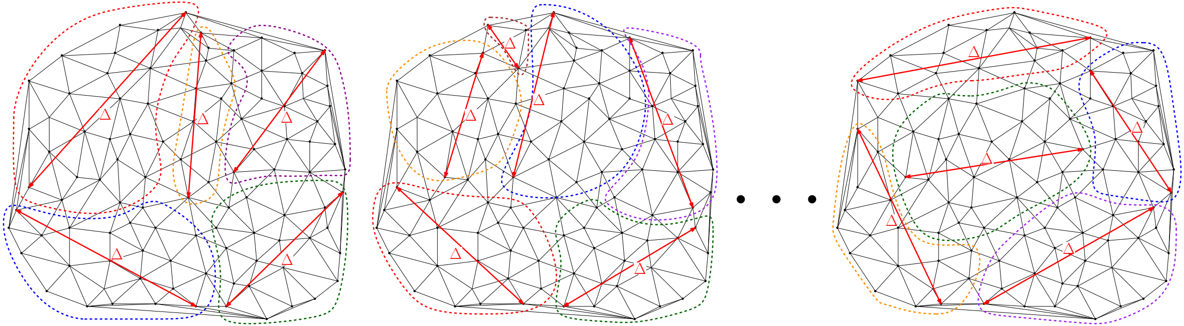

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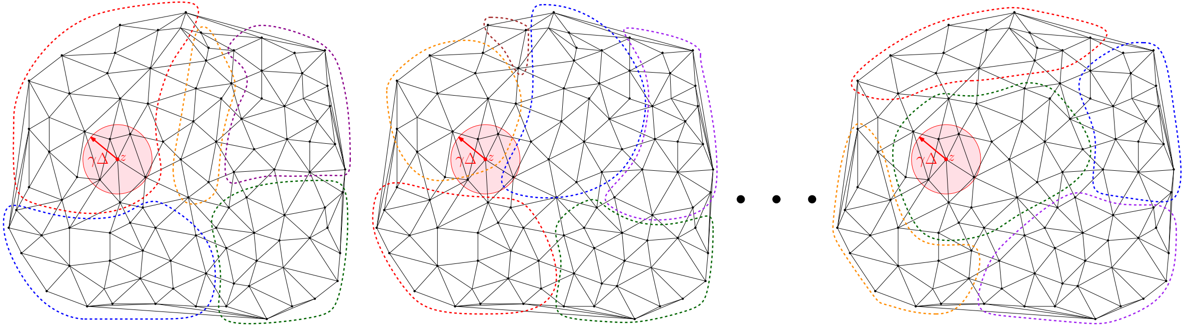
- $\forall$ small $0 \leq \gamma$, and $z \in V$, $\qquad \Pr[B(z, \gamma\Delta) \subseteq P(z)] \geq e^{-\beta\gamma} \approx 1 - \beta\gamma$



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Given a metric space $(X, d_X)$ (or a weight graph $G = (V, E, w)$).
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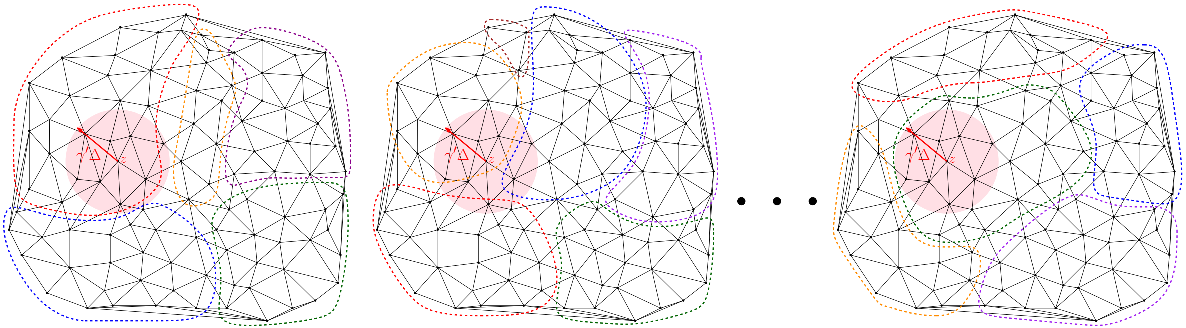

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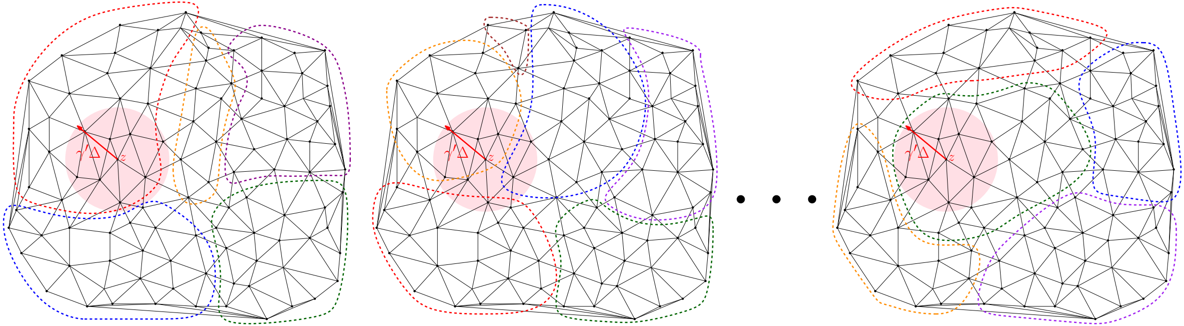

$G$ admits a $\beta$-padded decomposition **scheme**:

$\forall \Delta > 0$, $G$ admits $(\beta, \Delta)$-padded decomposition.

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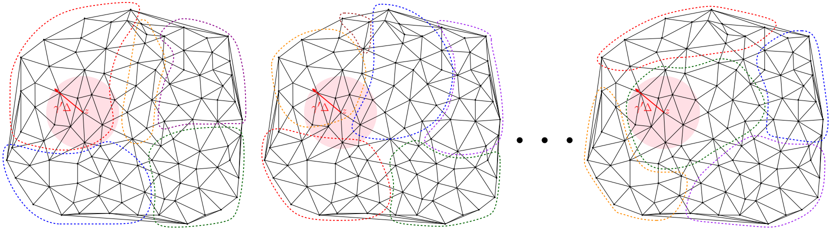

Note: $\Pr[B(z, \frac{1}{\beta} \cdot \Delta) \subseteq P(z)] \geq \Omega(1)$.

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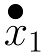

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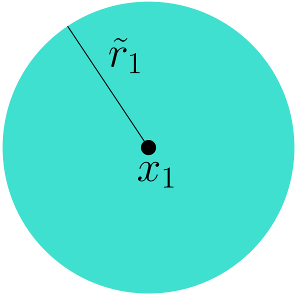

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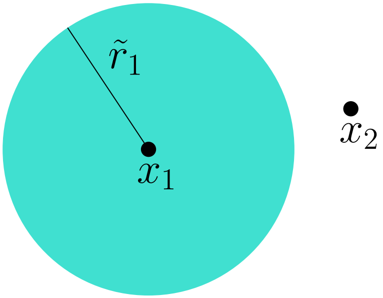

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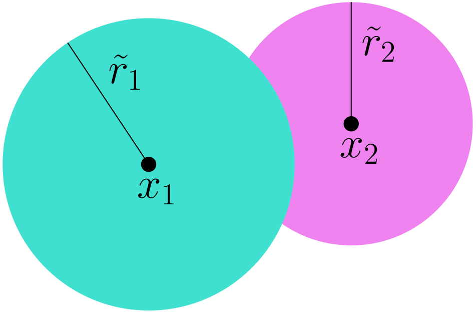

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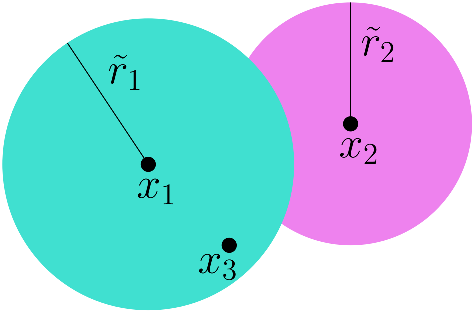

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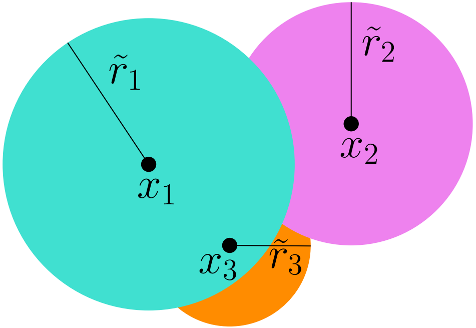

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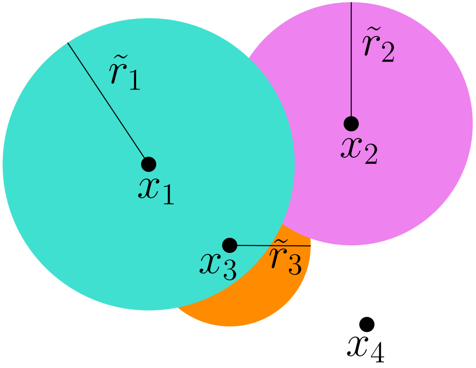

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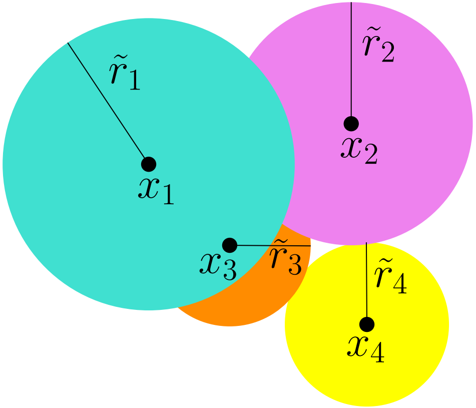

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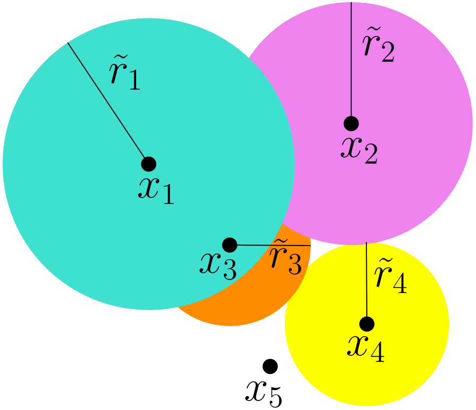

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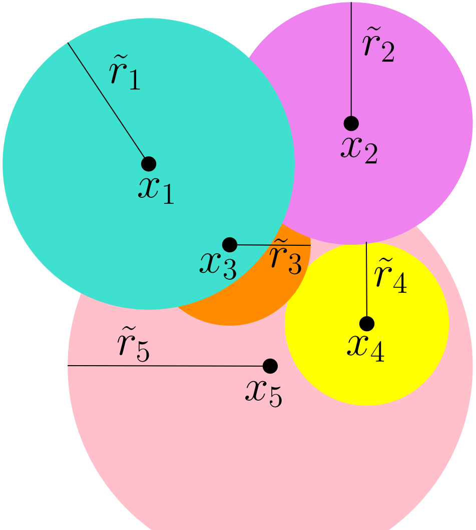

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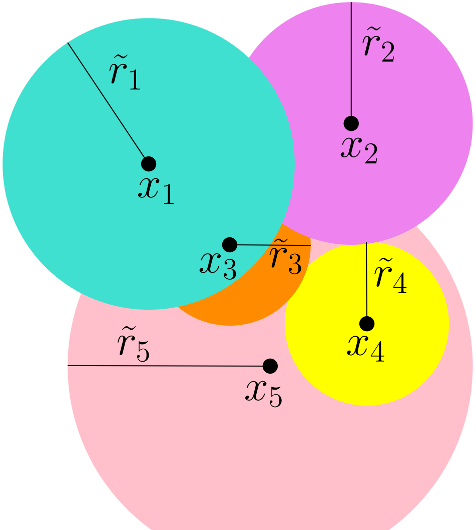

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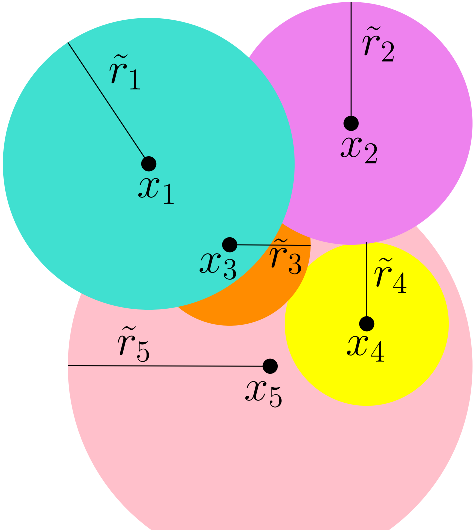


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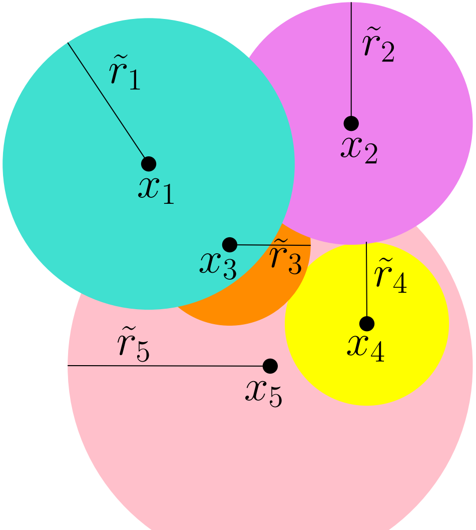
$\Rightarrow$ all clusters have diameter $\leq \Delta$.



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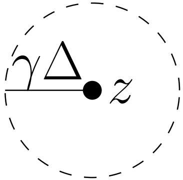


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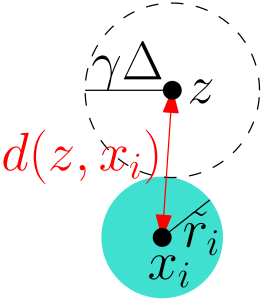


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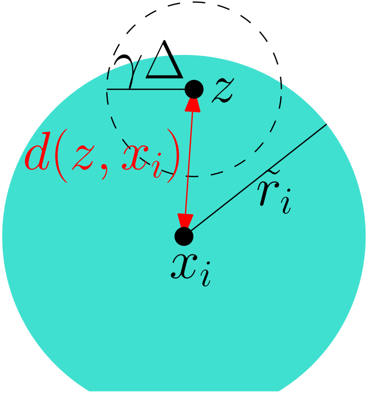


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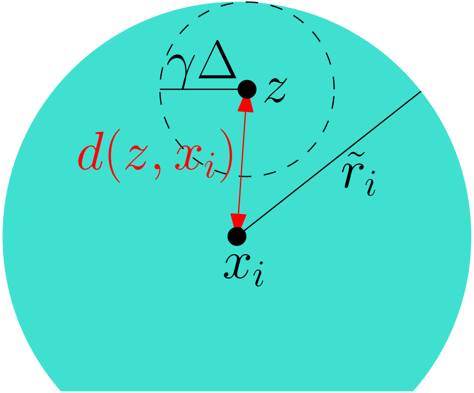

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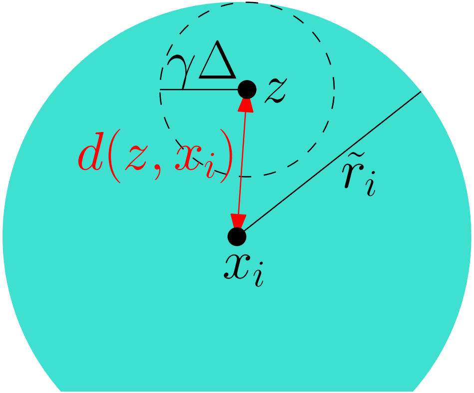


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By **Memorylessness**,

$$\Pr\left[B(z, \gamma\Delta) \subseteq C_i \mid B(z, \gamma\Delta) \cap C_i \neq \emptyset\right] \geq \Pr\left[\tilde{r}_i \geq 2\gamma\Delta\right] = e^{-\gamma \cdot 2c \log n}$$

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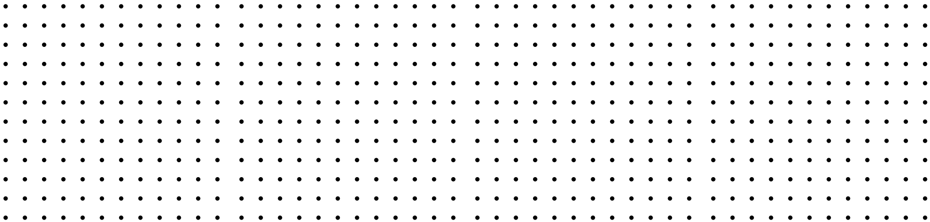

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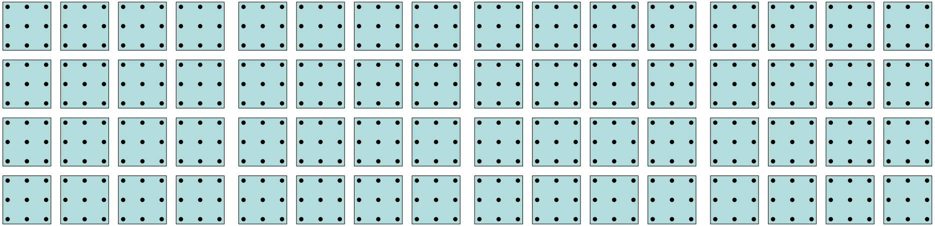

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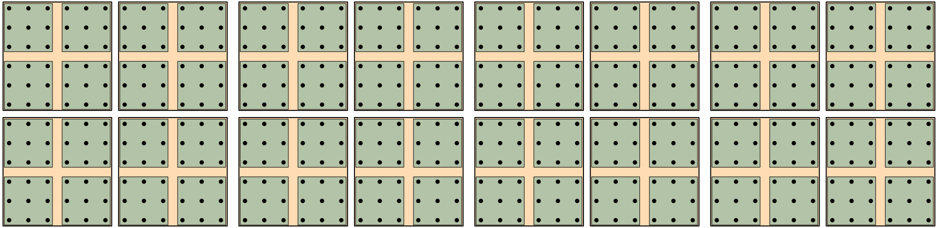

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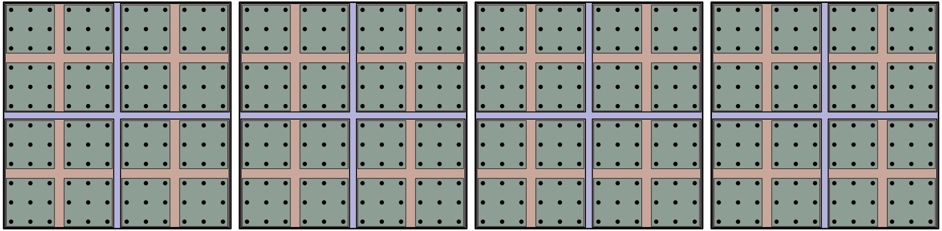

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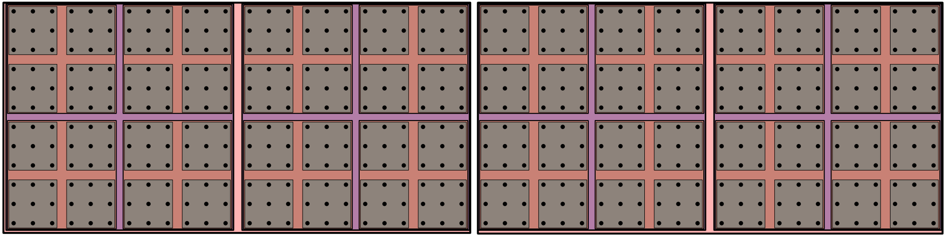

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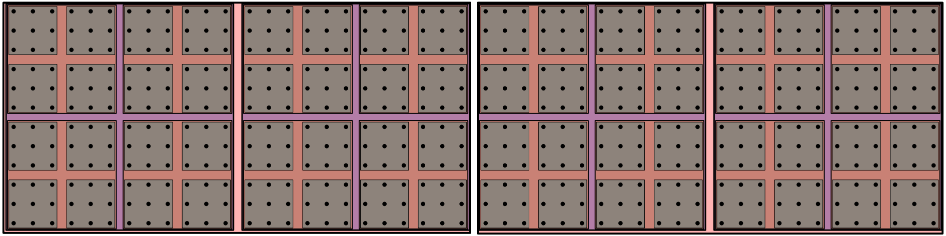

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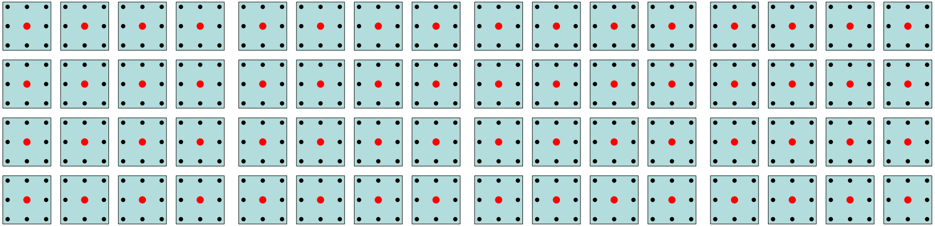

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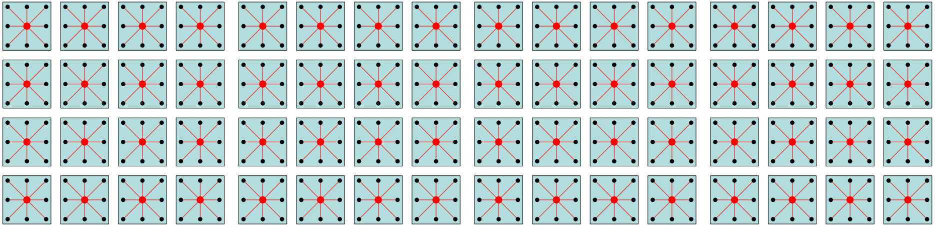

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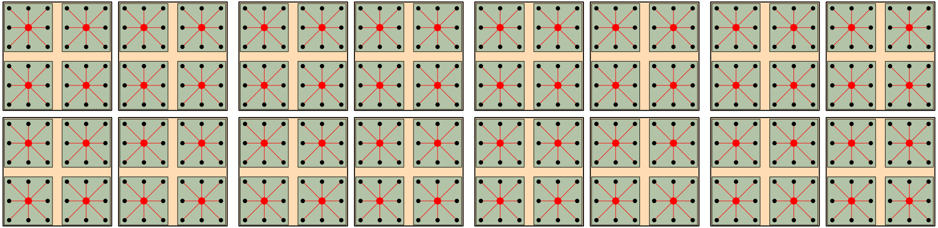

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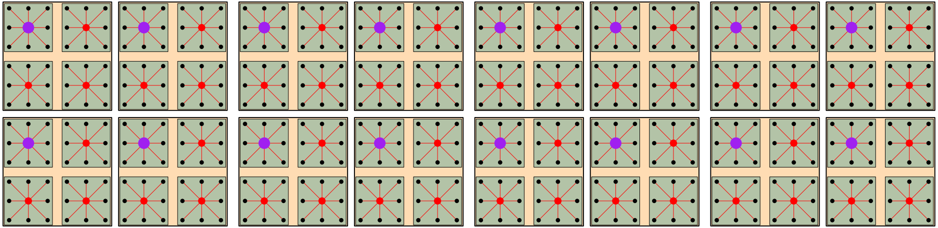

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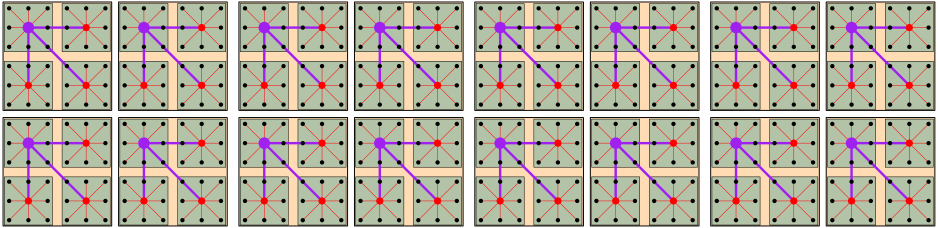

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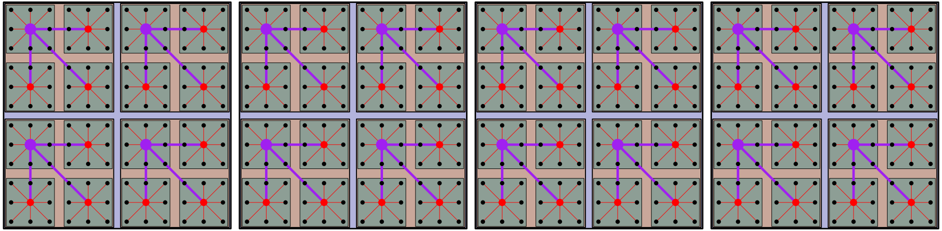

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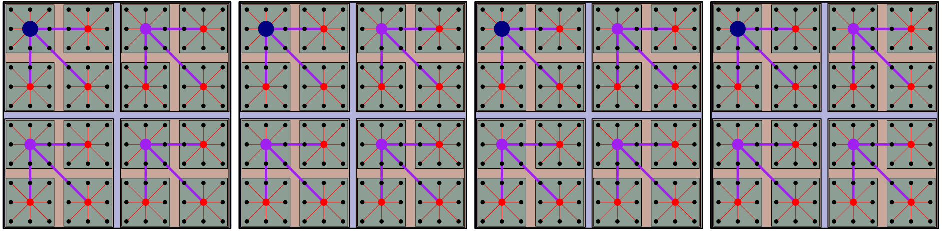

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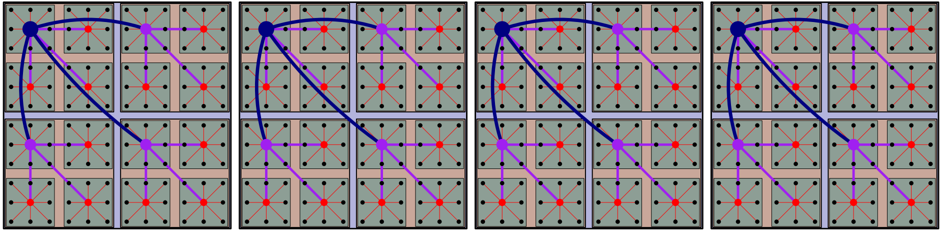

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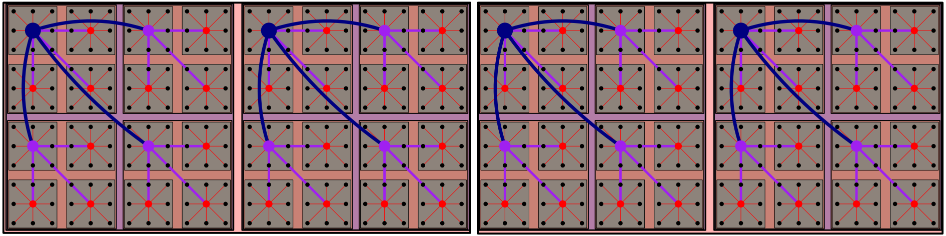

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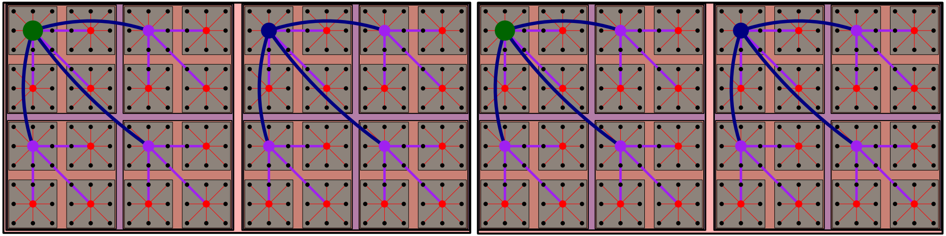

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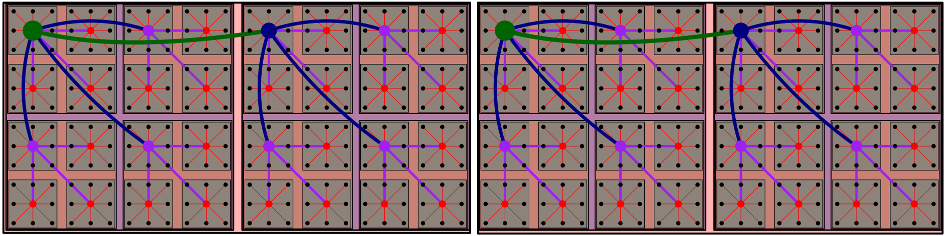

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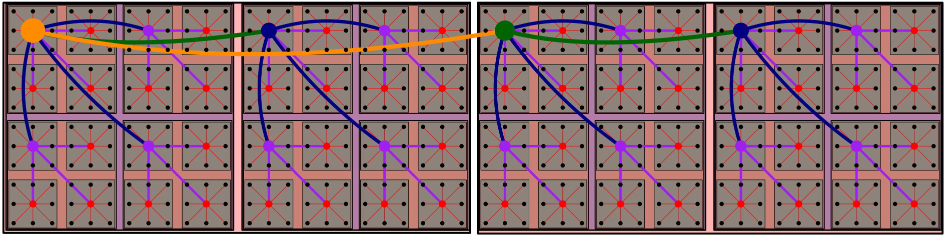

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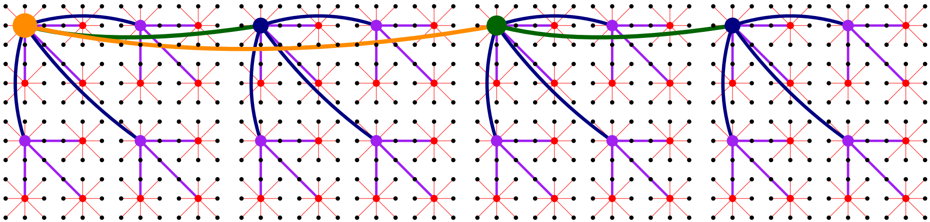

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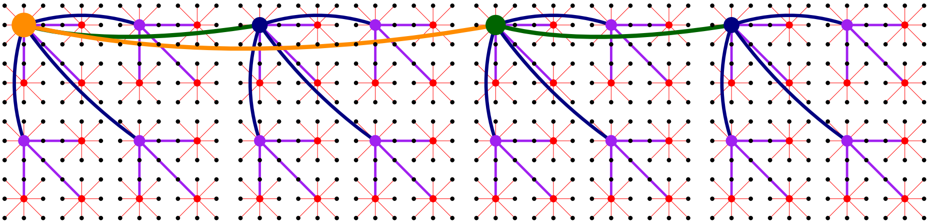

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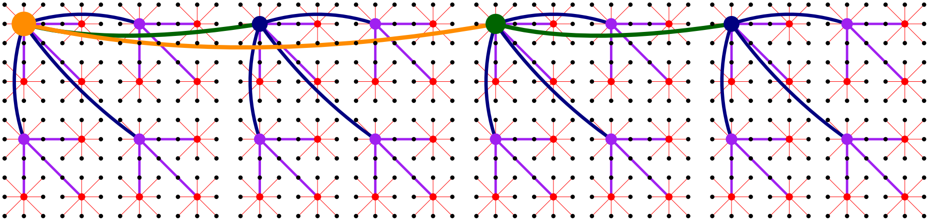

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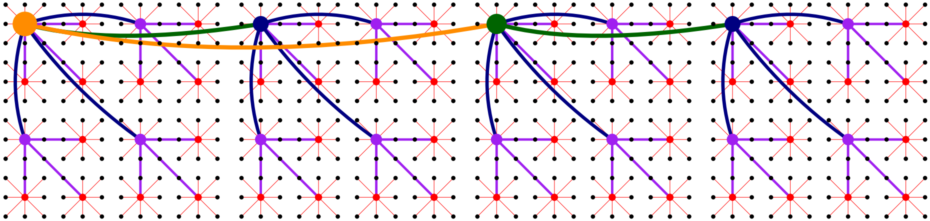

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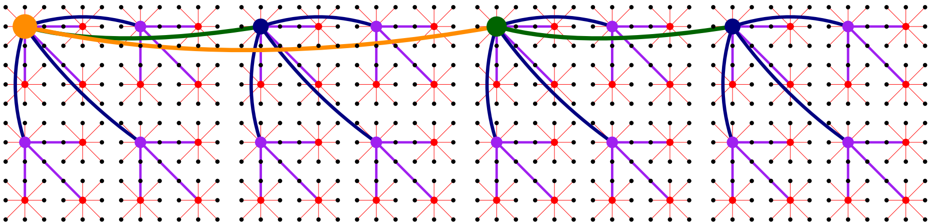

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For every $i \in [1, \log \Phi]$ sample an $\left(O(\log n), 2^i\right)$ padded decomposition $\mathcal{P}_i$.

**Observation 1:** For every $x, y \in X$, $d_X(x, y) \le d_T(x, y)$

**Observation 2:** If $\mathcal{P}_i(x) = \mathcal{P}_i(y)$, then $d_T(x, y) \le O(2^i)$

$d_T(x, y) = O(2^{i_{x,y}})$ where $i_{x,y}$ is the maximum index such that $\mathcal{P}_i(x) \neq \mathcal{P}_i(y)$.

$$\begin{aligned}
\mathbb{E}[d_T(x, y)] &\le \sum_{i=0}^{\log \Phi} O(2^i) \cdot \Pr\left[\mathcal{P}_i(x) \neq \mathcal{P}_i(y)\right] \\
&\le \sum_{i=0}^{\log \Phi} O(2^i) \cdot \frac{d_X(x, y)}{2^i} \cdot O(\log n) \\
&= O(\log^2 n) \cdot d_X(x, y) \ .
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The improvement is achieved by sampling the padded decomposition in various levels in a **correlated** fashion.

Specifically, the probability to cut $x, y$ at scale $\Delta$ is

$$\approx \frac{d_X(x,y)}{\Delta} \cdot \log \frac{|B(x, c \cdot 2^i)|}{|B(x, 2^i/c)|}$$

for some constant $c$, instead of $\approx \frac{d_X(x,y)}{\Delta} \cdot \log n$. Then the sum "telescopes".

# Outline of the talk

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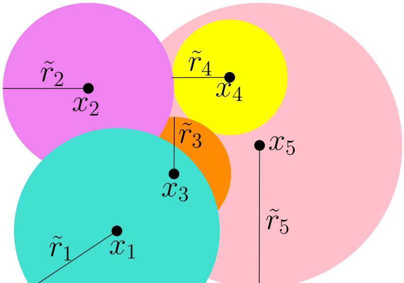

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Aspect ratio (a.k.a. spread) $\mathbf{\Phi} = \frac{\max_{x,y \in X} d_X(x,y)}{\min_{x,y \in X} d_X(x,y)}$

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This can also be done in an online fashion. Thus we obtain a deterministic embedding!

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Could we get deterministic distortion $\mathrm{poly}(n)$ for constant $d$?

# Online Padded Decomposition

## Definition (Padded Decomposition)

Given a metric space $(X, d_X)$ (or a weight graph $G = (V, E, w)$).
**Distribution** $\mathcal{D}$ over partitions of $G$ is $(\beta, \Delta)$-padded decomposition if:

- Every cluster $C \in \mathcal{P} \sim \mathcal{D}$ is $\Delta$-**bounded**.
- For every small $0 \leq \gamma$, and $z \in V$, $\quad \Pr[B(z, \gamma\Delta) \subseteq P(z)] \geq e^{-\beta\gamma}$ .

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# Online Padded Decomposition

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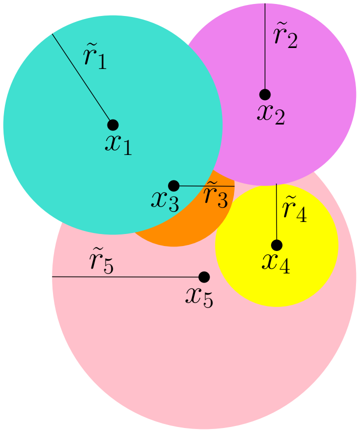


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## Question

Input: metric points in an online fashion from the **shortest path metric** of a planar graph. **Goal:** Sample from an $O(1)$-padded decomposition scheme.

**Crucially:** The planar graph is unknown!

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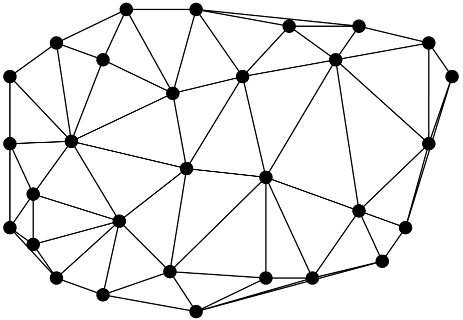

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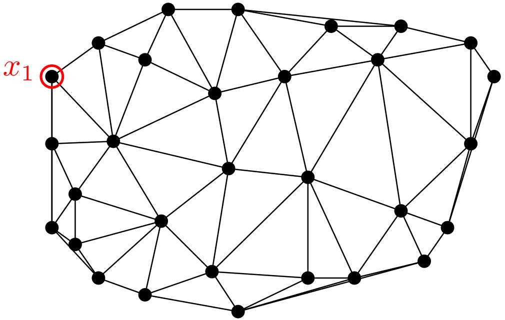

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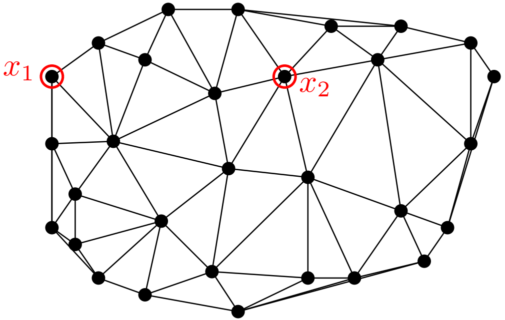


| | $x_1$ | $x_2$ |
|---|---|---|
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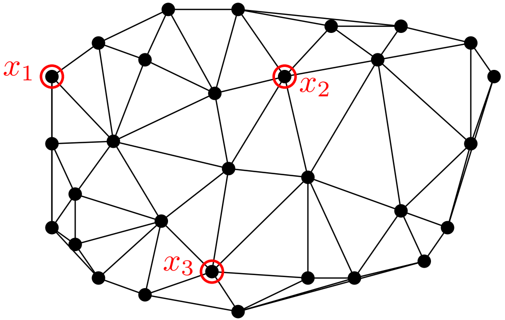


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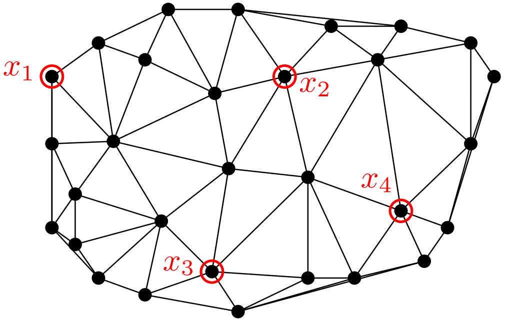


| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
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| $x_2$ | 3 | • | 2 | 2 |
| $x_3$ | 3 | 2 | • | 2 |
| $x_4$ | 4 | 2 | 2 | • |

# Outline of the talk

# Spanning Trees

**Theorem ([FRT04],[Bar04] Stochastic embedding into trees)**

*Every n-point metric space $(X, d)$ embeds into* ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

Suppose that we are given a graph $G = (V, E, w)$, could we sample a **spanning tree** of $G$ with expected distortion $O(\log n)$?

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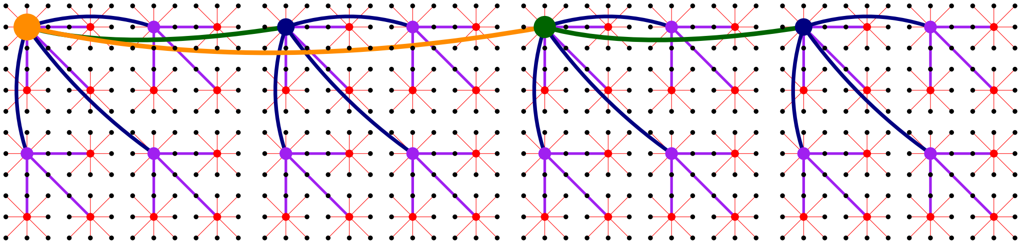
The construction we saw [Bar96], as well as others are not into subgraphs.



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**Theorem ([Abraham, Neiman 12] (improving over [AKPW95], [EEST05], [ABN08]))**

*Every n-vertex graph $G = (V, E, w)$ embeds into **distribution** $\mathcal{D}$ over it's **spanning trees** with **expected distortion** $\tilde{O}(\log n)$.*

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Construct embedding into spanning trees with expected distortion $O(\log n)$.

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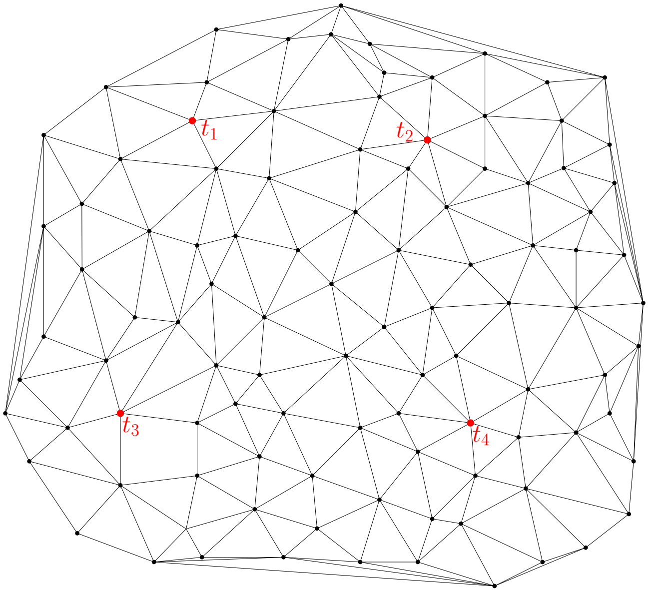
**Question**

Construct embedding into spanning trees with expected distortion $O(\log n)$.

We will see a recent, simple and elegant construction: [Becker, Emek, Ghaffari, Lenzen 24]. The expected distortion is $O(\log^3 n)$, and it is based on [MPX13].

# [Miller, Peng, Xu 2013] - clustering with exponential clocks



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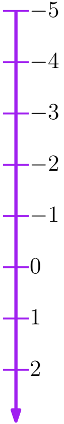


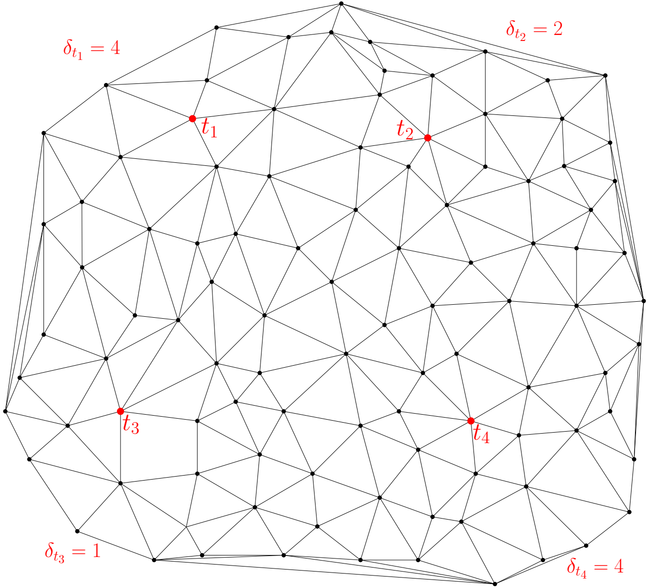

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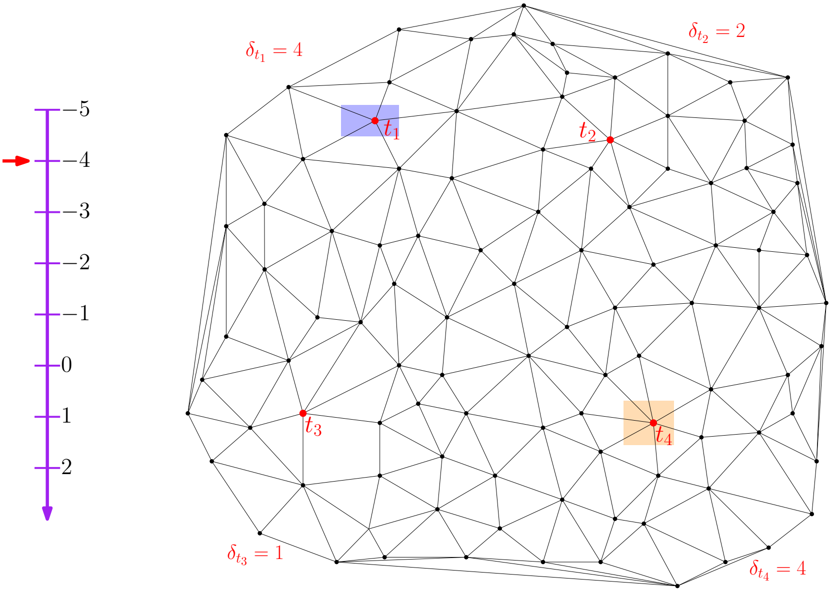

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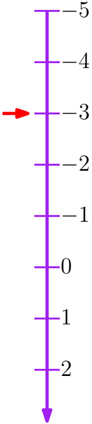


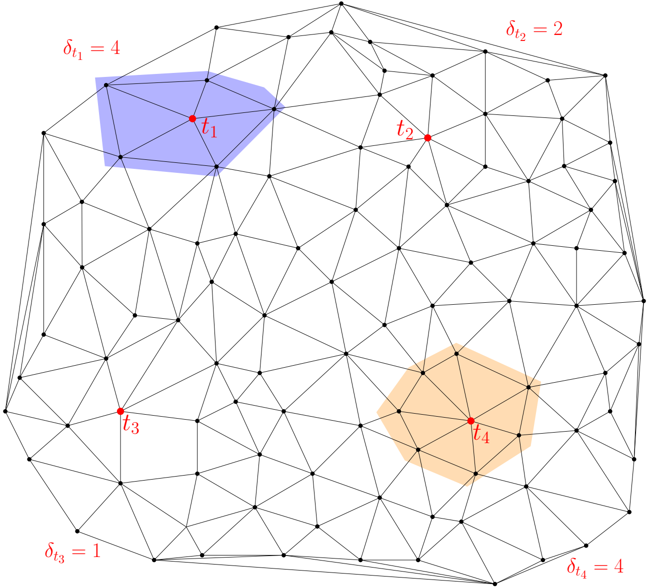

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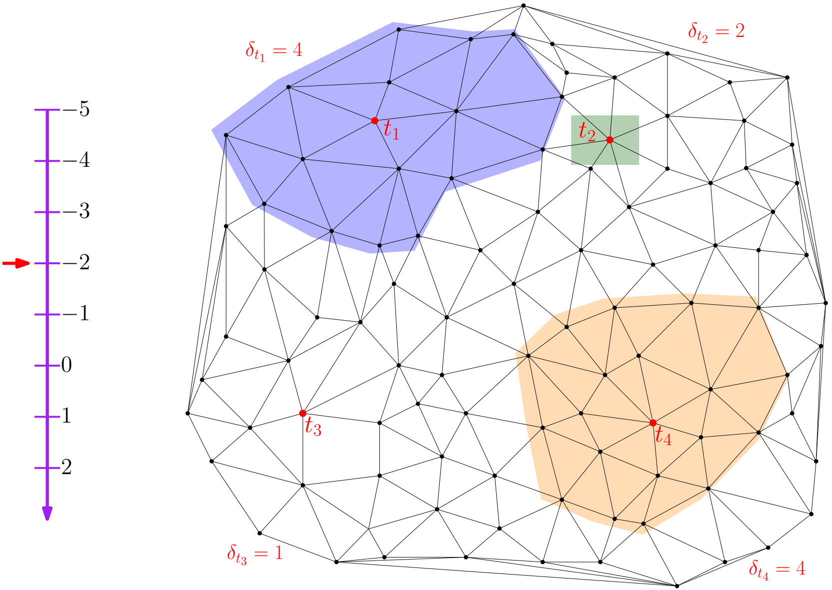

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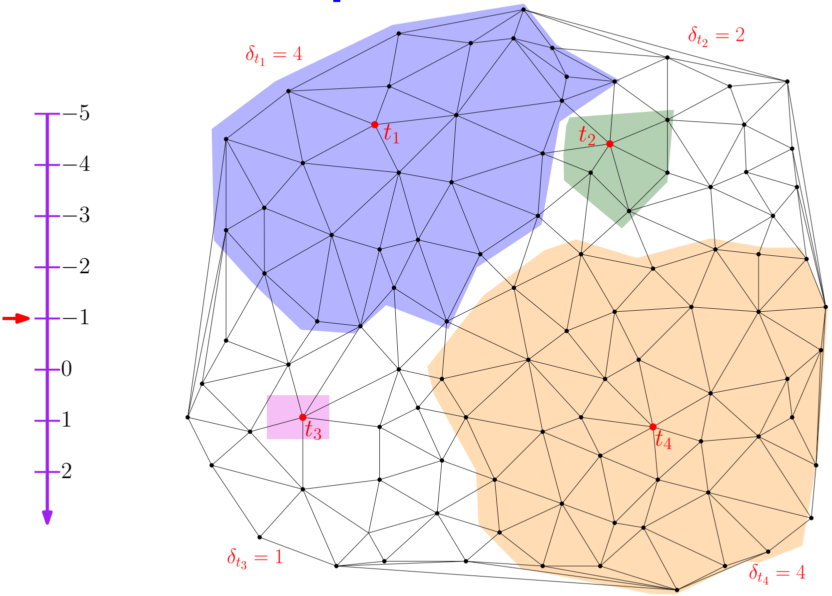

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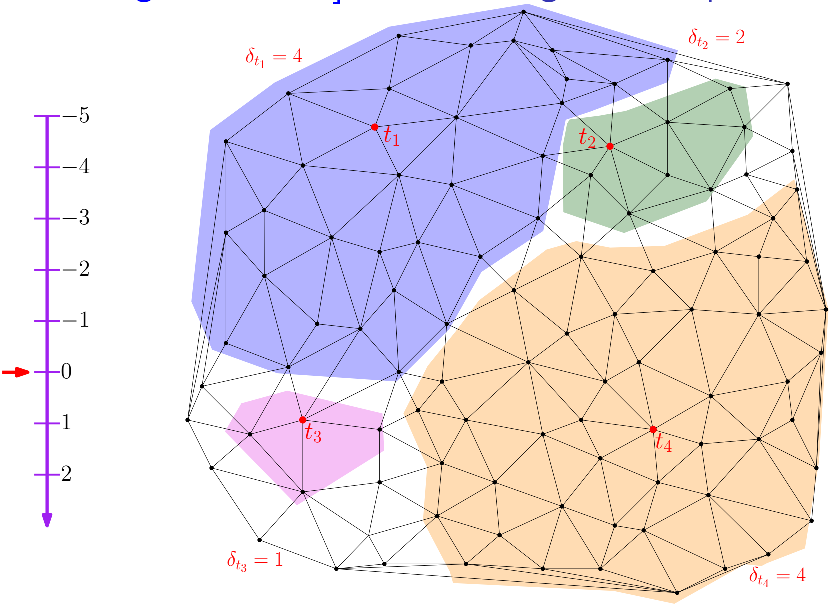

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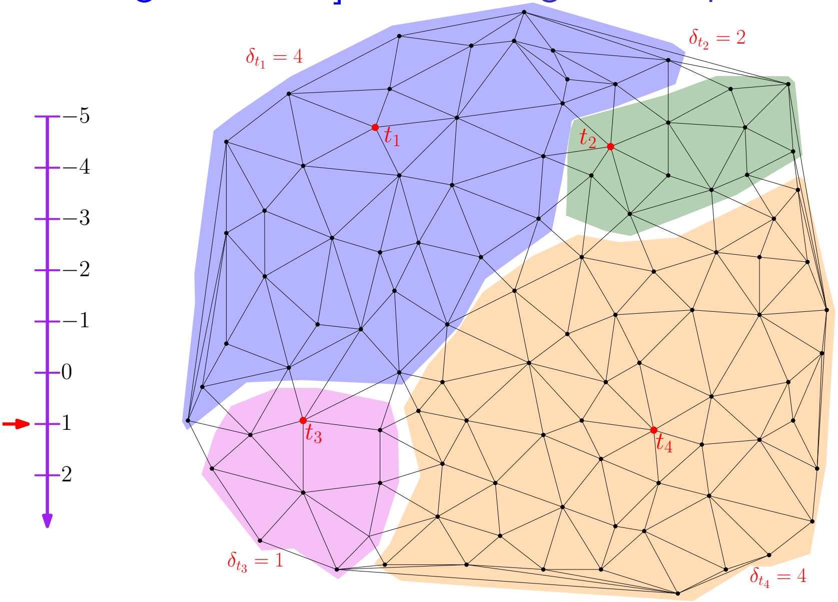

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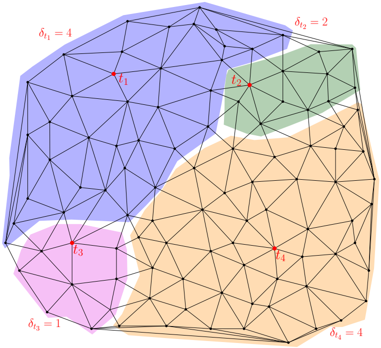


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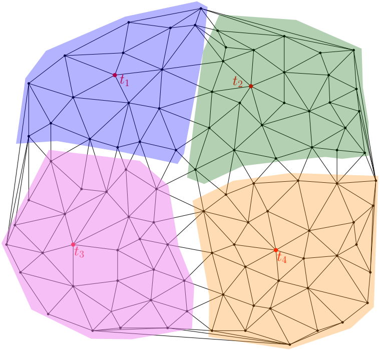


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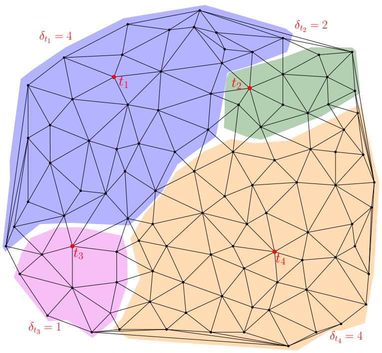


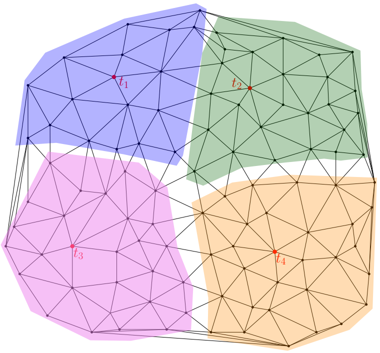


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[MPX13] produces a shifted Voronoi partition.

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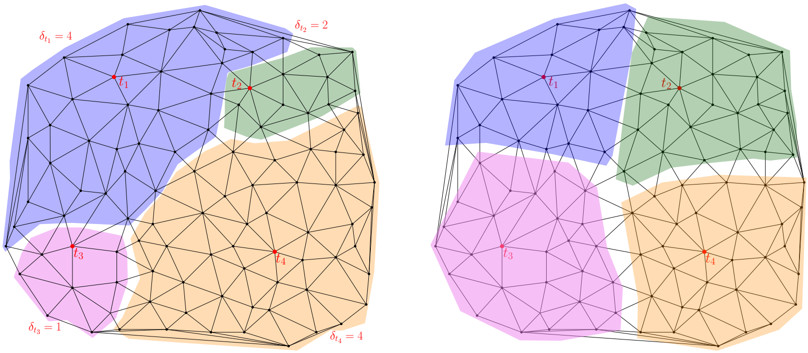


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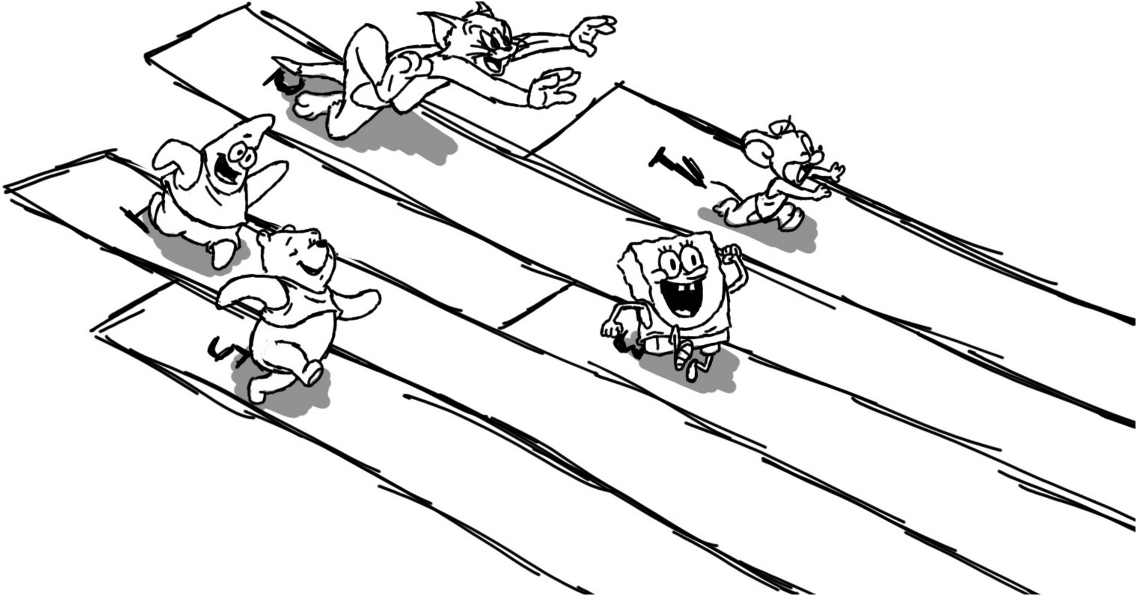
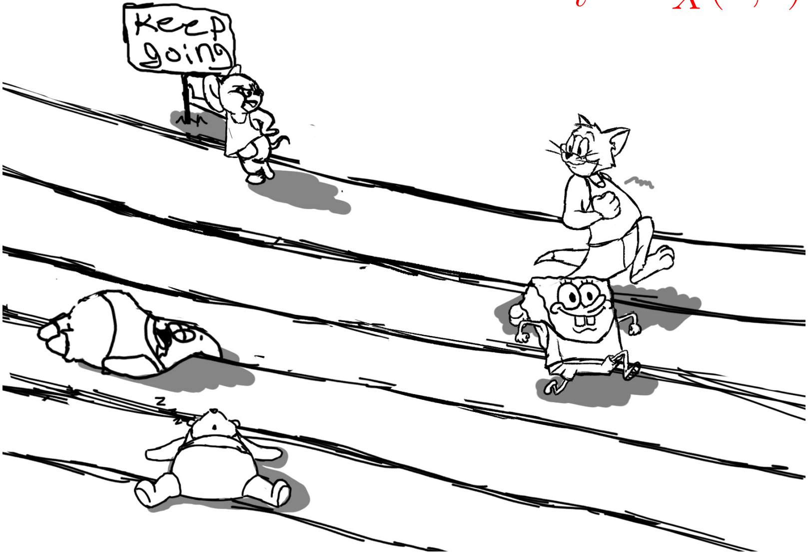
$\delta_t$ sampled i.i.d. using exponential distribution with parameter $\Theta(\frac{\Delta}{\log n})$.

Unfair race, different starting points: $\delta_t - d_X(v,t)$

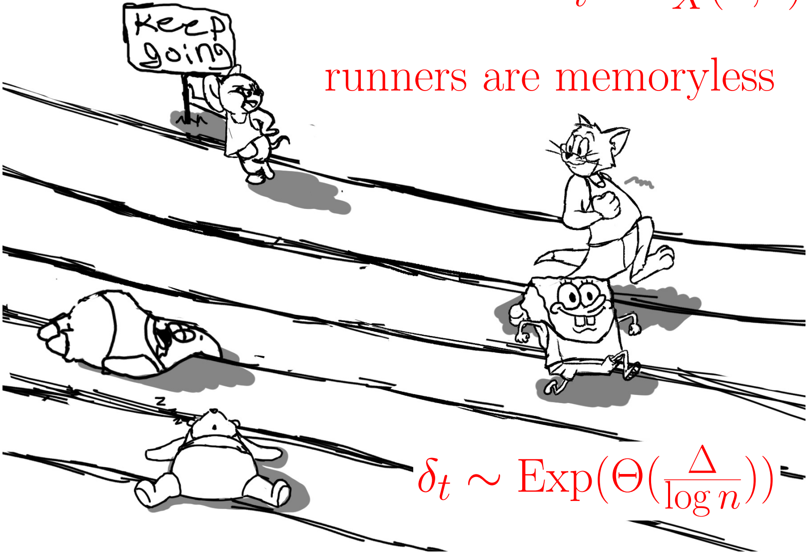


# Race of endurance: $\delta_t - d_X(v,t)$



Race of endurance: $\delta_t - d_X(v,t)$

runners are memoryless




$$\delta_t \sim \mathrm{Exp}(\Theta(\frac{\Delta}{\log n}))$$

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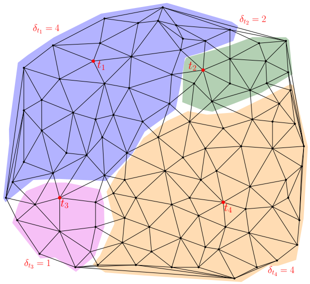

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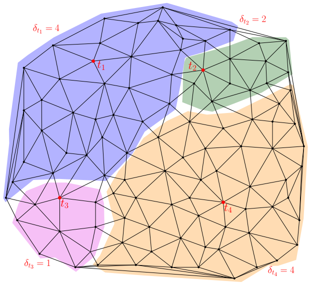


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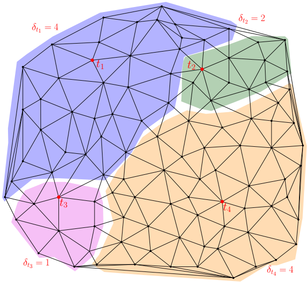


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**Theorem ([MPX13])**

*The algorithm produces an $(O(\log n), \Delta)$-padded decomposition.*

# Proof intuition - what makes it tick?

$v$ **joins** the cluster $C_t$ of the center $t$ **maximizing** $f_v$.

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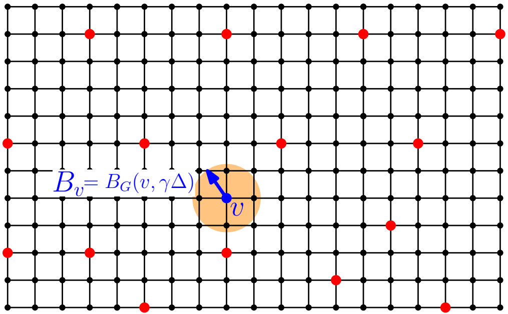

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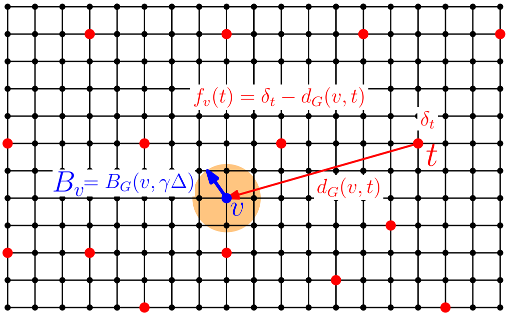

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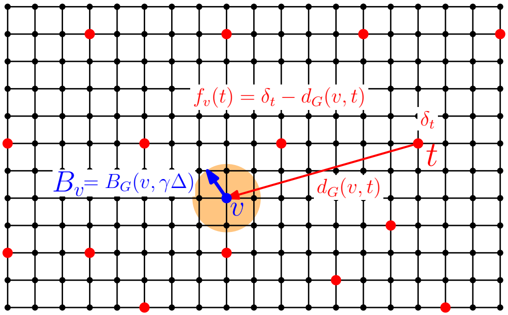


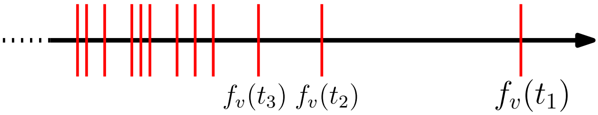


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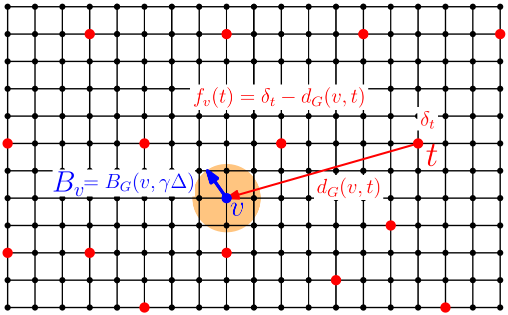


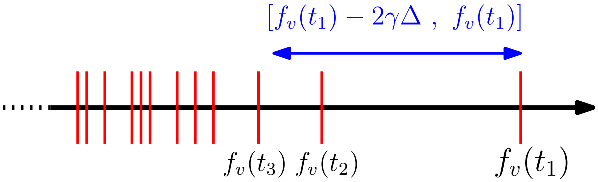


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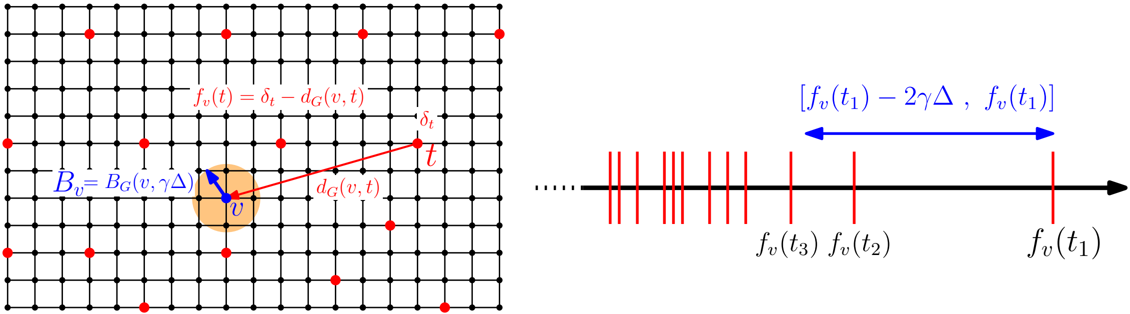


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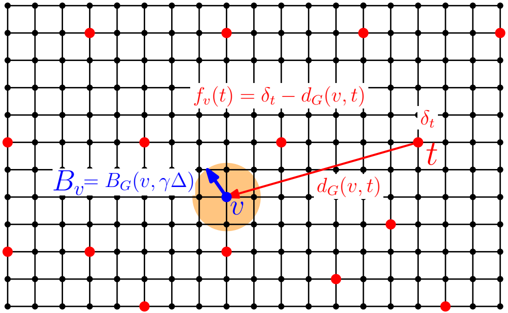


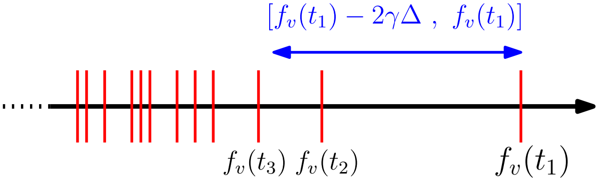


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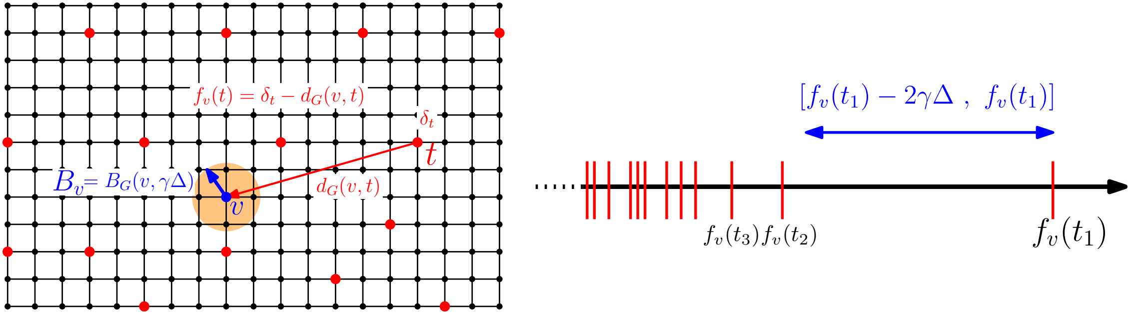


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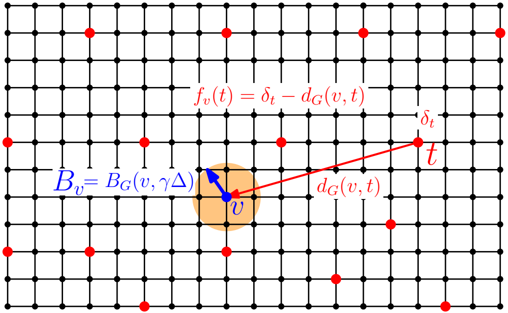


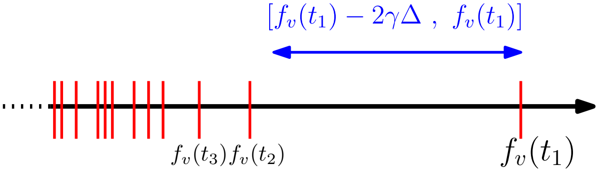


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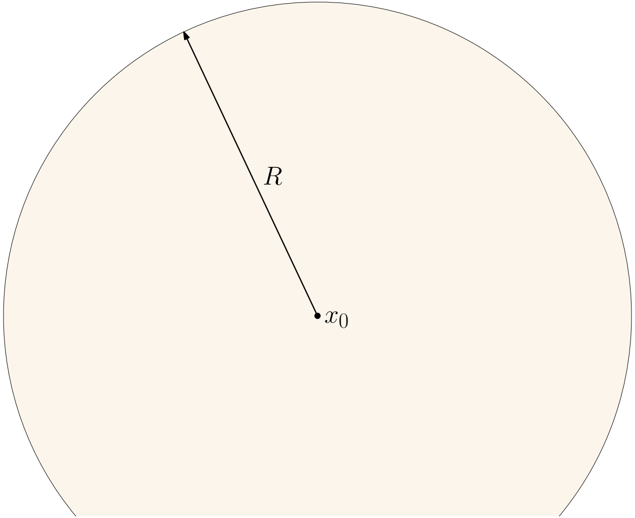
Spanning Tree Algorithm $(G, X, x_0, R)$

Assumption: $X \subseteq B(x_0, R)$



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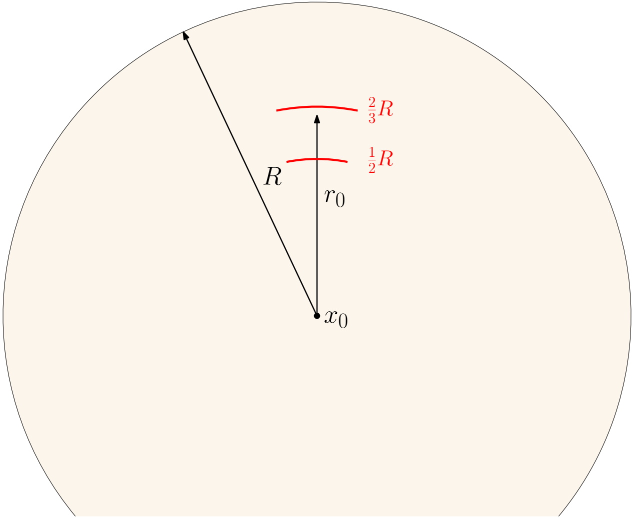

# [BEGL24] Stochastic Embedding into Spanning Trees

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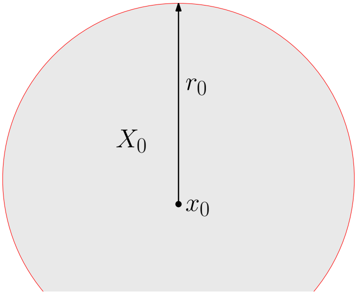

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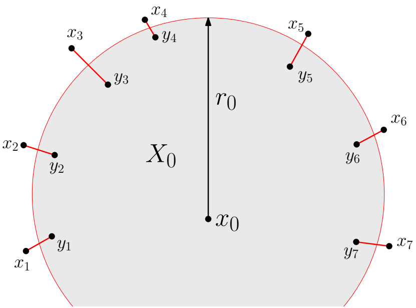

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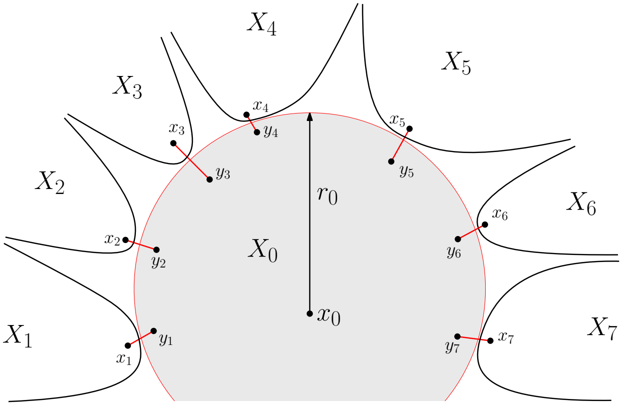

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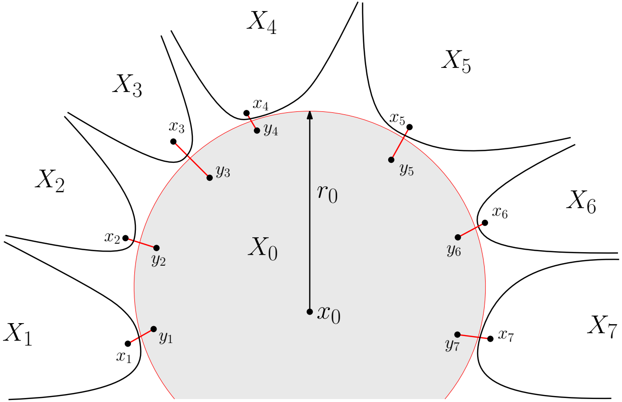


W.h.p. $\forall i, \delta_i \leq \epsilon R$

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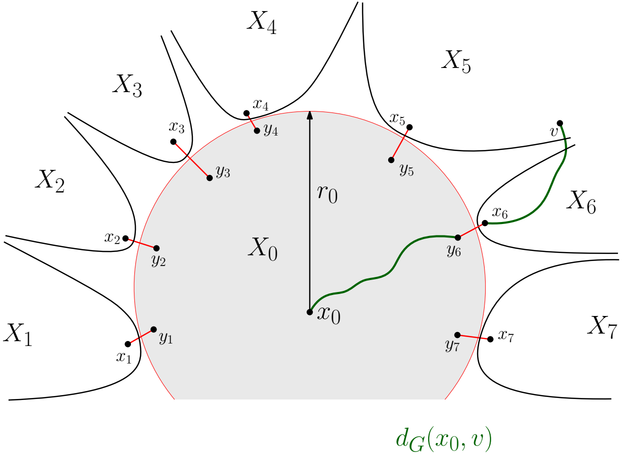


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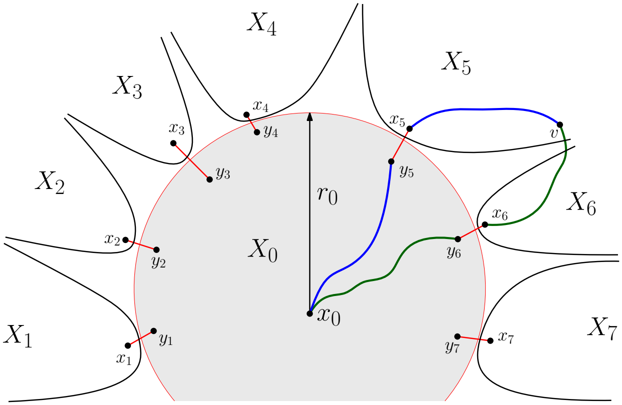


$d_G(x_0, x_5) + d_{G[X_5]}(x_5, v) \leq d_G(x_0, v) + \epsilon R$

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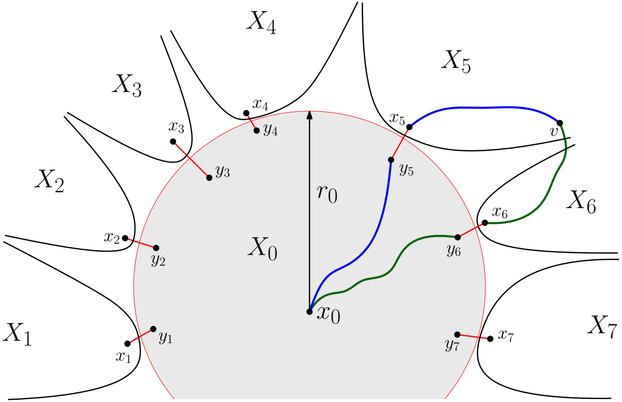


$$d_G(x_0, x_5) + d_{G[X_5]}(x_5, v) \le d_G(x_0, v) + \epsilon R$$
$$\le (1 + 2\epsilon) \cdot d_G(x_0, v)$$

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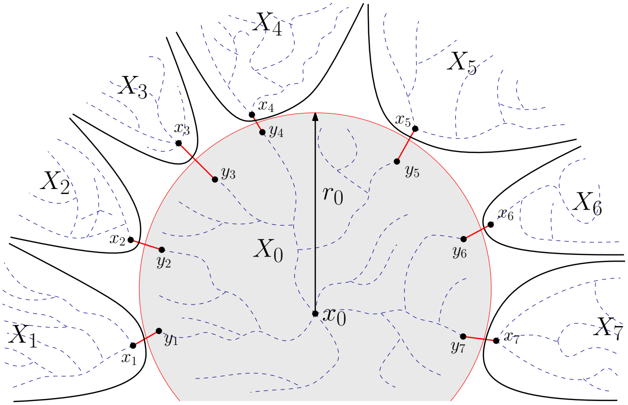

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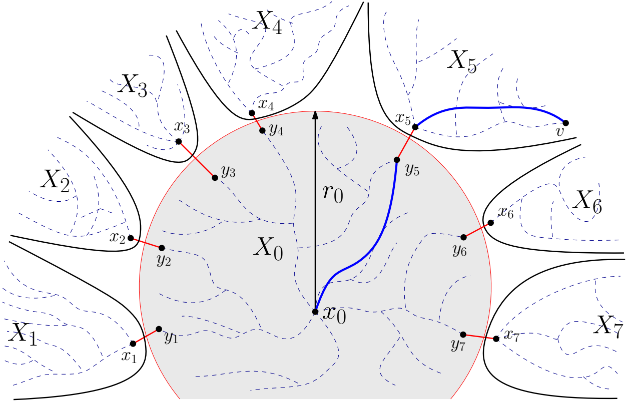


$d_G(x_0, x_5) + d_{G[X_5]}(x_5, v) \leq (1 + 2\epsilon) \cdot d_G(x_0, v)$

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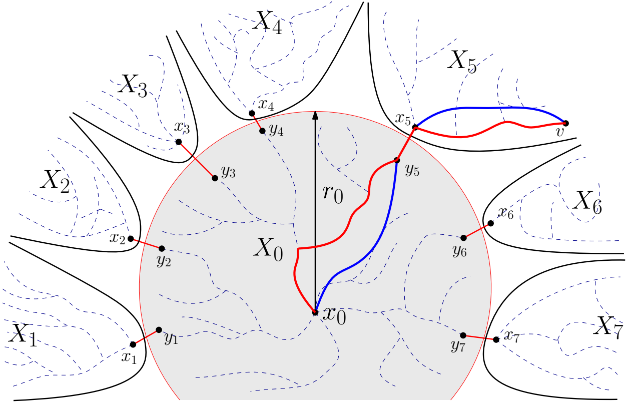


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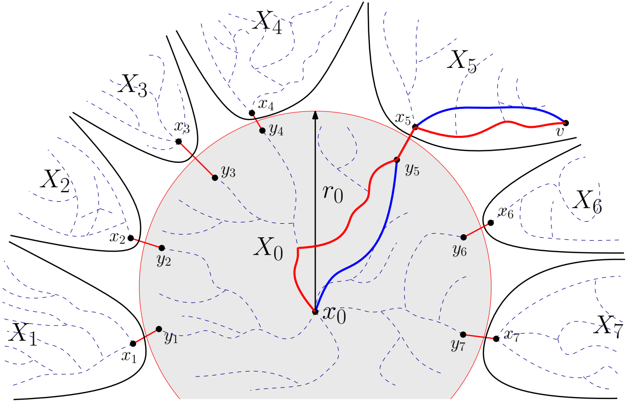


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$d_T(x_0, v) \leq (1 + 2\epsilon)^{O(\log n)} \cdot d_G(x_0, v) \leq 2 \cdot d_G(x_0, v)$

For $\epsilon = O(\frac{1}{\log n})$

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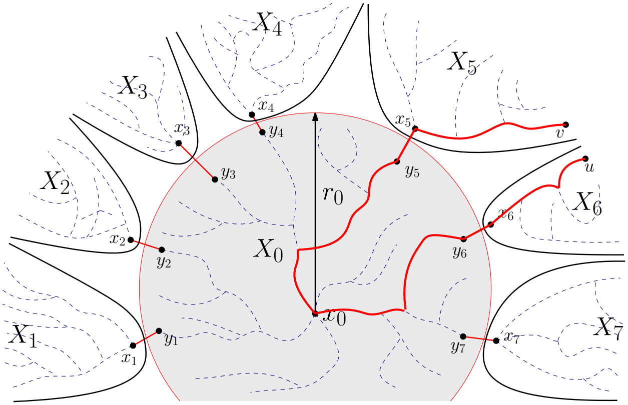


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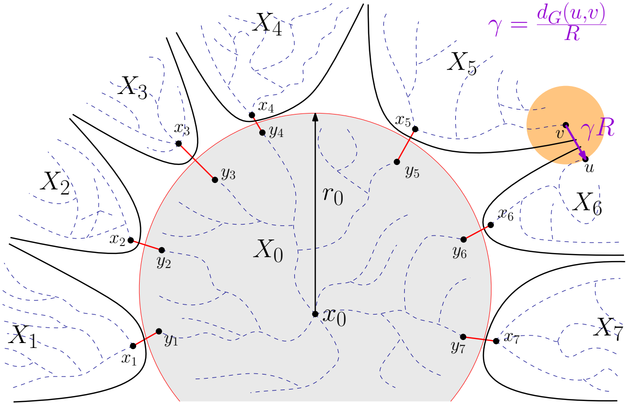
$\Pr[P_R(v) \neq P_R(u)] \geq$

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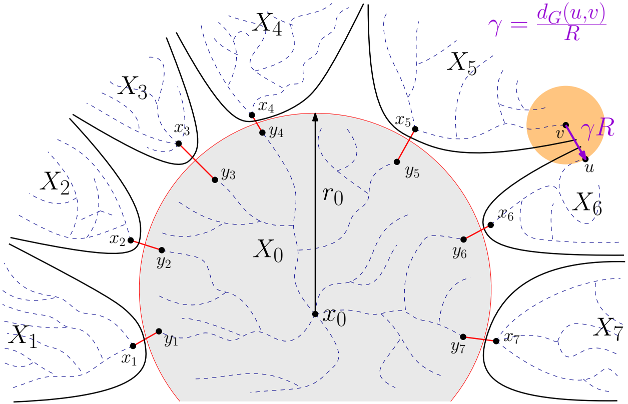


$\Pr[P_R(v) = P_R(u)] \geq \Pr[B(v, \gamma R) \subseteq P_R(v)]$

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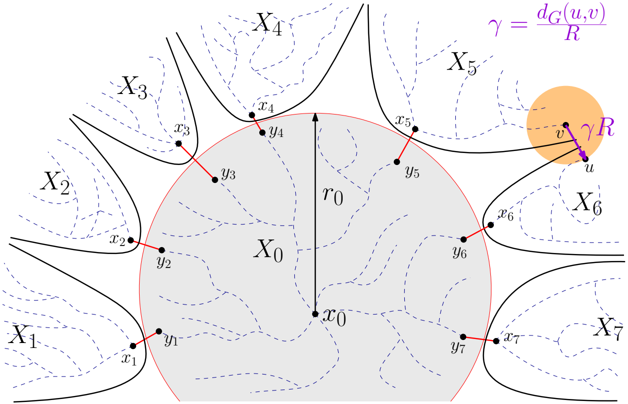


$$\Pr[P_R(v) = P_R(u)] \geq \Pr[B(v, \gamma R) \subseteq P_R(v)]$$

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$$\approx 1 - \log^2 n \cdot \frac{d_G(u,v)}{R}$$

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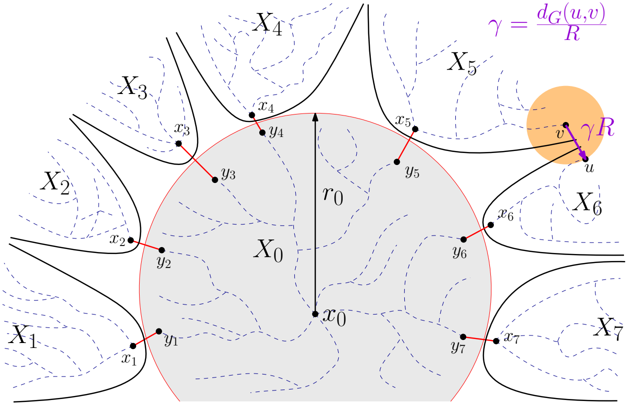
$$\mathbb{E}[d_T(x,y)]$$
$$\leq \sum_{i=0}^{\log \Phi} O(2^i) \cdot \Pr\left[\mathcal{P}_i(x) \neq \mathcal{P}_i(y)\right]$$
$$\leq \sum_{i=0}^{\log \Phi} O(2^i) \cdot \frac{d_X(x,y)}{2^i} \cdot \boldsymbol{O(\log^2 n)}$$
$$= \boldsymbol{O(\log^3 n)} \cdot d_X(x,y)\ .$$

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# Outline of the talk

# Special Graph Families

**Theorem ([FRT04],[Bar04] Stochastic embedding into trees)**

*Every n-point metric space $(X, d)$ embeds into* ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

# Special Graph Families

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Tight!

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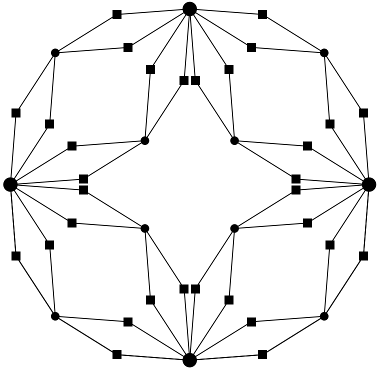
Try **special graph families**!

# Special Graph Families

**Special graph families**:

- Planar.



# Special Graph Families

**Special graph families**:

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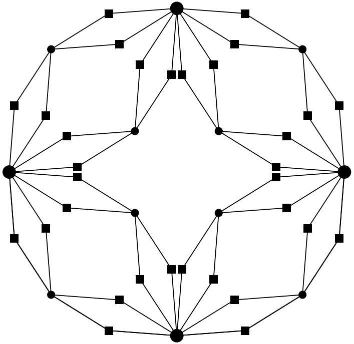
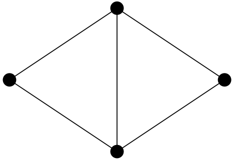
- Excluding a fixed minor.



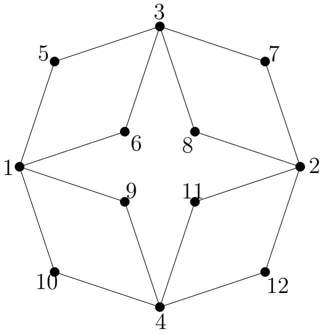


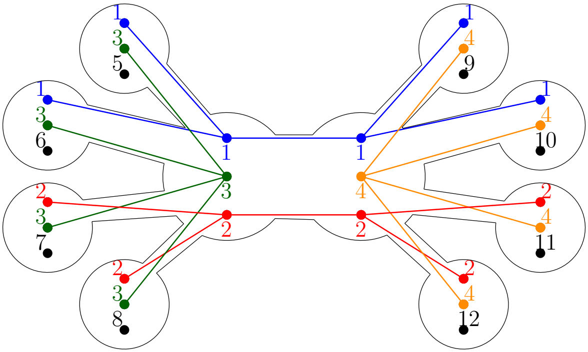
Excluded as a minor

# Special Graph Families

**Special graph families**:

- Planar.
- Excluding a fixed minor.
- Bounded treewidth.



# Planar graphs into treewidth graphs

**Theorem ([FRT04], improving [Bartal 96+98])**

*Every $n$-point metric space $(X, d)$ embeds into distribution $\mathcal{D}$ over trees with expected distortion $O(\log n)$.*

Tight!

Planar graphs into trees, could we do **better**?

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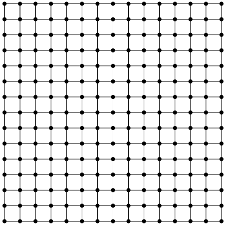
[AKPW95]: [FRT04] is tight already for the $n \times n$ grid graph!



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**Theorem ([Chakrabarti, Jaffe, Lee, Vincent 08] Stochastic Lower Bound)**

*$\forall \boldsymbol{t}$, $\exists$ n-vertex **planar** graph $G = (V, E, w)$ such that **every stochastic embedding** into treewidth-$\boldsymbol{t}$ graphs incur distortion $\Omega(\log n)$.*

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**Applications** to: PTAS for **Vehicle routing**, EPTAS for metric **$\rho$-dominating** set, and metric $\rho$-isolated set.

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**Theorem ([Cohen-Addad, F., Klein, Le 20] Minor lower bound)**

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A version of this embedding (called Ramsey type and clan) been used to obtain QPTAS for the $\rho$-dominating/isolated set problems.

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Questions:

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- For planar graphs we have treewidth $\min\{\mathrm{poly}(\epsilon^{-1}), O(\epsilon^{-1}(\log\log n)^2)\}$. Can we get $O(\epsilon^{-1})$?

# Embedding into treewidth graphs with multiplicative distortion

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**Theorem ([Carroll, Goel 04] Stochastic Lower Bound 2)**

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**Theorem ([Chang, Cohen-Addad, Conroy, Le, Pilipczuk, Pilipczuk 25])**

$\forall \epsilon \in (0,1)$, *every n-point* ***planar*** *graph embeds into* ***distribution*** *over graphs with* ***treewidth*** $O(\epsilon^{-1} \cdot \log^3 n)$ *with* ***expected distortion*** $1+\epsilon$.

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**Conjecture [CCCLPP25]**

$1+\epsilon$ **expected distortion** from **planar** graphs into **treewidth** $O(\epsilon^{-1} \cdot \log n)$ graphs.

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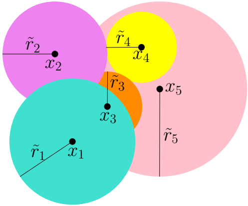
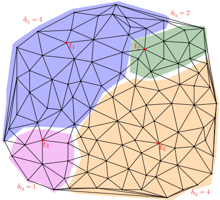
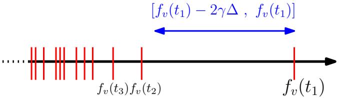
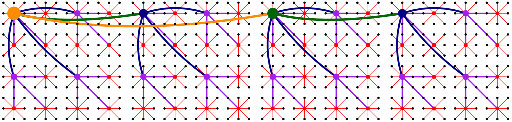
- More?

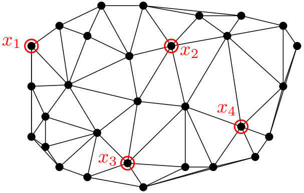
# Thank you!



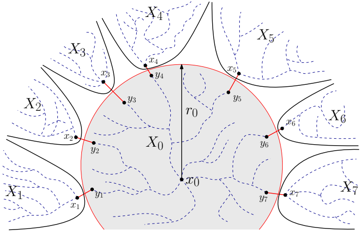








| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | • | 3 | 3 | 4 |
| $x_2$ | 3 | • | 2 | 2 |
| $x_3$ | 3 | 2 | • | 2 |
| $x_4$ | 4 | 2 | 2 | • |




# Questions?

# Outline of the talk